PART B

Unit - VIII

Difference Equations & Z-transforms

8.1 Introduction

We have earlier discussed (*Unit-VI*) finite differences and some related topics. We introduce ***Difference Equations*** based on the concept of finite differences, whose general / complete solution can be obtained in a manner analogous to the method of solving linear differential equations with constant coefficients.

We discuss ***Z-transforms*** in detail and also the solution of difference equations using Z-transforms.

8.2 Difference Equations

Recalling the basic definition of finite (*forward*) difference of $f(x)$: $\Delta f(x) = f(x+h) - f(x)$, we have in general

$$\Delta y_n = y_{n+1} - y_n$$

Further, $\Delta^2 y_n = \Delta(\Delta y_n) = y_{n+2} - 2y_{n+1} + y_n$ etc.

Suppose that,

$$\Delta y_n = 1 \ (say), \quad \Delta^2 y_n + \Delta y_n = 0 \ (say)$$

we obtain equations of the form

$$y_{n+1} - y_n = 1, \quad y_{n+2} - y_{n+1} = 0$$

In fact, these type of equations are referred to as **Difference Equations.**

Definition : A **Difference Equation** is a relationship between the differences of an unknown function (*dependent variable y*) at several values of the independent variable. (*argument x*)

Difference Equations are also called *Recurrence relations*

An equation of the form,

$$a_r y_{n+r} + a_{r-1} y_{n+r-1} + a_{r-2} y_{n+r-2} + \cdots + a_2 y_{n+2} + a_1 y_{n+1} + a_0 y_n = \phi(n)$$

where n take the values 0, 1, 2, 3, . . . and a_r, a_{r-1}, . . . a_0 are all constants is called a *linear difference equation of order r.*

In other words we can say that a difference equation is a relationship interms of the values y_{n+r}, y_{n+r-1}, . . . y_{n+1}, y_n.

Finding the sequence y_n constitutes a solution of the difference equation.

It may be noted that the general solution of a differnce equation contains arbitrary constants equal to the order of the difference equation. (*Analogous to O.D.E*)

We proceed to discuss Z-transforms.

8.3 Z-transforms

8.31 Introduction

We are acquainted with Laplace transforms and Fourier transforms whose basic definition is in the form of a definite integral in which the integrand is involved with two parameters. The resulting integral whenever it exists will be a function of a single parameter.

Z - transforms operates on the sequences of functions of a single variable defined for non negative integral values of the variable. This transform has number of properties similar to that of Laplace transforms.

Difference equations arises in situations with the data consisting of only a set of values of an unknown function (*discrete values*). Just as Laplace transforms and Fourier transforms serves as a tool to solve some types of ordinary and partial differential equations, *Z transforms serves as a tool to solve difference equations.*

Z - transforms play an important role in the analysis and representations of discrete time linear shift invariant systems. It has applications in control system of engineering and also in some advanced statistical problems.

8.32 Definition

If $u_n = f(n)$ defined for all $n = 0, 1, 2, 3, \ldots$ and $u_n = 0$ for $n < 0$ then the ***Z-transform*** of u_n denoted by $Z_T(u_n)$ is defined by

$$Z_T(u_n) = \sum_{n=0}^{\infty} u_n z^{-n} \qquad \ldots (1)$$

Whenever the series on the R.H.S of (1) converges it will be a function of z and we write,

$$Z_T(u_n) = \bar{u}(z) \qquad \ldots (2)$$

Further (2) can be written in the equivalent form

$$Z_T^{-1}[\bar{u}(z)] = u_n \qquad \ldots (3)$$

This is called the **Inverse *Z-transform.***

Note : *Notation* $Z(u_n)$, $Z^{-1}[\bar{u}(z)]$ *is also used.*

Notation $U(z)$ *for* $\bar{u}(z)$ *is also used.*

■ **Property** : $Z_T(n^k) = -z\frac{d}{dz} Z_T(n^{k-1})$ *where k is a positive integer.*

Proof : Consider R.H.S

That is, $-z\frac{d}{dz} Z_T(n^{k-1}) = -z\frac{d}{dz}\sum_{n=0}^{\infty} n^{k-1} z^{-n}$

$$= -z\sum_{n=0}^{\infty} n^{k-1}(-n) z^{-n-1}$$

$$= \sum_{n=0}^{\infty} n^k z^{-n} = Z_T(n^k) = \text{L.H.S}$$

Thus we have proved that, $Z_T(n^k) = -z\frac{d}{dz} Z_T(n^{k-1})$

8.33 Z - transform of some standard functions

1. $Z_T(k^n)$ 2. $Z_T(1)$ 3. $Z_T(n)$ 4. $Z_T(n^2)$ 5. $Z_T(n^3)$

1. By the definition, $Z_T(k^n) = \sum_{n=0}^{\infty} k^n z^{-n} = \sum_{n=0}^{\infty}(k/z)^n$

$$= 1 + (k/z) + (k/z)^2 + \cdots$$

The series on the R.H.S is a geometric series of the form $1 + r + r^2 + \cdots$ whose sum to infinity is $1/1 - r$ where $r = k/z$

$\therefore \quad Z_T(k^n) = \frac{1}{1-(k/z)}$

Thus $\quad Z_T(k^n) = \frac{z}{z-k}$

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2. Putting $k = 1$ in the previous result we have $k^n = 1$

Thus $\quad Z_T(1) = \frac{z}{z-1}$

Remark : *The result can easily be established independently also.*

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Results (3), (4), (5) are established by using the property

$$Z_T(n^k) = -z\frac{d}{dz}Z_T(n^{k-1})$$

We obtain the required Z-transforms by taking $k = 1$, $k = 2$, $k = 3$

3. When $k = 1$, $Z_T(n) = -z\frac{d}{dz}Z_T(n^0) = -z\frac{d}{dz}Z_T(1)$

i.e., $$Z_T(n) = -z\frac{d}{dz}\left(\frac{z}{z-1}\right) = -z\left\{\frac{(z-1)-z}{(z-1)^2}\right\}$$

Thus $$\boldsymbol{Z_T(n) = \frac{z}{(z-1)^2}}$$

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4. When $k = 2$, $Z_T(n^2) = -z\frac{d}{dz}Z_T(n)$

i.e., $$Z_T(n^2) = -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$= -z\left\{\frac{(z-1)^2 - z\cdot 2(z-1)}{(z-1)^4}\right\}$$

$$= -z(z-1)\left\{\frac{z-1-2z}{(z-1)^4}\right\} = \frac{-z(-z-1)}{(z-1)^3}$$

Thus $$\boldsymbol{Z_T(n^2) = \frac{z(z+1)}{(z-1)^3} = \frac{z^2+z}{(z-1)^3}}$$

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5. When $k = 3$, $Z_T(n^3) = -z\frac{d}{dz}(n^2)$

i.e., $$Z_T(n^3) = -z\frac{d}{dz}\left[\frac{z^2+z}{(z-1)^3}\right]$$

$$= -z\left\{\frac{(z-1)^3(2z+1)-(z^2+z)\cdot 3(z-1)^2}{(z-1)^6}\right\}$$

$$= -z(z-1)^2\left\{\frac{(z-1)(2z+1)-3(z^2+z)}{(z-1)^6}\right\}$$

$$Z_T(n^3) = \frac{-z(-z^2-4z-1)}{(z-1)^4} = \frac{z^3+4z^2+z}{(z-1)^4}$$

Thus $\quad \mathbf{Z_T(n^3) = \frac{z^3+4z^2+z}{(z-1)^4}}$

8.4 Linearity property

Statement : *If u_n and v_n be any two discrete valued functions then*

$Z_T(c_1 u_n + c_2 v_n) = c_1 Z_T(u_n) + c_2 Z_T(v_n)$ *where c_1, c_2 are constants.*

Proof : We have by the definition,

$$Z_T(c_1 u_n + c_2 v_n) = \sum_{n=0}^{\infty} (c_1 u_n + c_2 v_n) z^{-n}$$

$$= c_1 \sum_{n=0}^{\infty} u_n z^{-n} + c_2 \sum_{n=0}^{\infty} v_n z^{-n}$$

Thus $\quad \mathbf{Z_T(c_1 u_n + c_2 v_n) = c_1 Z_T(u_n) + c_2 Z_T(v_n)}$

8.5 Damping rule (property)

Statement : *If $Z_T(u_n) = \overline{u}(z)$ then*

(i) $Z_T(k^n u_n) = \overline{u}(z/k)$ $\qquad$ **(ii)** $Z_T(k^{-n} u_n) = \overline{u}(kz)$

Proof : **(i)** $Z_T(k^n u_n) = \sum_{n=0}^{\infty} (k^n u_n) z^{-n}$

$$= \sum_{0}^{\infty} u_n (z/k)^{-n} = \overline{u}(z/k)$$

Thus $\quad \mathbf{Z_T(k^n u_n) = \overline{u}(z/k)}$

(ii) $\quad Z_T(k^{-n} u_n) = \sum_{n=0}^{\infty} (k^{-n} u_n) z^{-n}$

$$= \sum_{0}^{\infty} u_n (kz)^{-n} = \overline{u}(kz)$$

Thus $\quad \mathbf{Z_T(k^{-n} u_n) = \overline{u}(kz)}$

8.51 Some applications of damping rule

We have already obtained $Z_T(n)$, $Z_T(n^2)$, $Z_T(n^3)$ and we can obtain $Z_T(k^n n)$, $Z_T(k^n n^2)$, $Z_T(k^n n^3)$ by using damping rule.

(i) $Z_T(k^n n) = \left| Z_T(n) \right|_{z \to (z/k)}$ where $Z_T(n) = \dfrac{z}{(z-1)^2}$

$$\therefore \quad Z_T(k^n n) = \frac{(z/k)}{(z/k-1)^2}$$

Thus $\quad \mathbf{Z_T(k^n u_n) = \dfrac{kz}{(z-k)^2}}$

(ii) We have $Z_T(n^2) = \dfrac{z^2+z}{(z-1)^3}$

$$\therefore \quad Z_T(k^n n^2) = \left| Z_T(n^2) \right|_{z \to (z/k)} = \frac{(z/k)^2 + (z/k)}{(z/k-1)^3}$$

Thus $\quad \mathbf{Z_T(k^n n^2) = \dfrac{kz^2 + k^2 z}{(z-k)^3}}$

(iii) We have $Z_T(n^3) = \dfrac{z^3+4z^2+z}{(z-1)^4}$

$$\therefore \quad Z_T(k^n n^3) = \left| Z_T(n^3) \right|_{z \to (z/k)} = \frac{(z/k)^3 + 4(z/k)^2 + (z/k)}{(z/k-1)^4}$$

Thus $\quad \mathbf{Z_T(k^n n^2) = \dfrac{kz^3 + 4k^2 z^2 + k^3 z}{(z-k)^4}}$

List of standard Z - transforms

1. $Z_T(1) = \dfrac{z}{z-1}$	2. $Z_T(k^n) = \dfrac{z}{z-k}$
3. $Z_T(n) = \dfrac{z}{(z-1)^2}$	4. $Z_T(k^n n) = \dfrac{kz}{(z-k)^2}$
5. $Z_T(n^2) = \dfrac{z^2+z}{(z-1)^3}$	6. $Z_T(k^n n^2) = \dfrac{kz^2+k^2 z}{(z-k)^3}$
7. $Z_T(n^3) = \dfrac{z^3+4z^2+z}{(z-1)^4}$	8. $Z_T(k^n n^3) = \dfrac{kz^3+4k^2 z^2+k^3 z}{(z-k)^4}$

8.6 Shifting rule (property)

1. ***Right shifting rule***

If $Z_T(u_n) = \overline{u}(z)$ *then* $Z_T(u_{n-k}) = z^{-k}\,\overline{u}(z)$ *where* $k > 0$

Proof : By the definition,

$$Z_T(u_{n-k}) = \sum_{n=0}^{\infty} u_{n-k}\, z^{-n}$$

Since $u_n = 0$ for $n < 0$ in the general context, we have $u_{n-k} = 0$ for $n = 0, 1, 2, \cdots (k-1)$

$$\therefore \quad Z_T(u_{n-k}) = \sum_{n=k}^{\infty} u_{n-k}\, z^{-n}$$

$$= u_0 z^{-k} + u_1 z^{-(k+1)} + u_2 z^{-(k+2)} + \cdots$$

$$= z^{-k}(u_0 + u_1 z^{-1} + u_2 z^{-2} + \cdots)$$

$$= z^{-k} \sum_{n=0}^{\infty} u_n z^{-n} = z^{-k}\, \overline{u}(z)$$

Thus $\mathbf{Z_T(u_{n-k}) = z^{-k}\, \overline{u}(z)}$

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2. ***Left shifting rule***

$$Z_T(u_{n+k}) = z^k \left[\overline{u}(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \cdots - u_{k-1} z^{-(k-1)} \right]$$

or

$$Z_T(u_{n+k}) = z^k \left[\overline{u}(z) - \sum_{r=0}^{k-1} u_r\, z^{-r} \right]$$

Proof : $Z_T(u_{n+k}) = \sum_{n=0}^{\infty} u_{n+k} z^{-n} = z^k \sum_{n=0}^{\infty} u_{n+k}\, z^{-(n+k)}$

i.e., $Z_T(u_{n+k}) = z^k \left[u_k z^{-k} + u_{k+1} z^{-(k+1)} + u_{k+2} z^{-(k+2)} + \cdots \right]$

$$= z^k \Big[\Big(u_0 + u_1 z^{-1} + u_2 z^{-2} + \cdots + u_{k-1} z^{-(k-1)} + u_k z^{-k} + u_{k+1} z^{-(k+1)} + \cdots \Big)$$

$$- \left(u_0 + u_1 z^{-1} + u_2 z^{-2} + \cdots + u_{k-1} z^{-(k-1)} \right) \Big]$$

$$= z^k \left[\sum_{n=0}^{\infty} u_n z^{-n} - \left(u_0 + u_1 z^{-1} + u_2 z^{-2} + \cdots + u_{k-1} z^{-(k-1)} \right) \right]$$

Thus $\quad \mathbf{Z_T(u_{n+k}) = z^k \left[\bar{u}(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \cdots - u_{k-1} z^{-(k-1)} \right]}$

Note : *Some particular cases.*

$$Z_T\,(u_{n+1}) = z\left[\bar{u}(z) - u_0\right]$$

$$Z_T\,(u_{n+2}) = z^2\left[\bar{u}(z) - u_0 - u_1 z^{-1}\right]$$

$$Z_T\,(u_{n+3}) = z^3\left[\bar{u}(z) - u_0 - u_1 z^{-1} - u_2 z^{-2}\right] \text{ etc.}$$

WORKED PROBLEMS

1. *Find the Z - transforms of the following .*

(i) e^{-an} **(ii)** $e^{-an}\, n$ **(iii)** $e^{-an}\, n^2$

>> (i) $e^{-an} = (e^{-a})^n = k^n$ (*say*) where $k = e^{-a}$

We have, $Z_T(k^n) = \dfrac{z}{z-k}$

Thus $\quad \mathbf{Z_T(e^{-an}) = \dfrac{z}{z-e^{-a}}}$

(ii) We have $Z_T(n) = \dfrac{z}{(z-1)^2}$

$$\therefore \qquad Z_T(k^n\, n) = \left\{ \frac{z}{(z-1)^2} \right\}_{z \to (z/k)} = \frac{(z/k)}{(z/k-1)^2}$$

i.e., $\quad Z_T(k^n\, n) = \dfrac{kz}{(z-k)^2}$

Taking $k = e^{-a}$ we obtain,

$$\mathbf{Z_T(e^{-an}\, n) = \frac{e^{-a}\, z}{(z-e^{-a})^2}}$$

(iii) We have $Z_T(n^2) = \dfrac{z^2+z}{(z-1)^3}$

$$\therefore \quad Z_T(k^n n^2) = \left\{\frac{z^2+z}{(z-1)^3}\right\}_{z \to (z/k)} = \frac{(z/k)^2+(z/k)}{(z/k-1)^3}$$

i.e., $$Z_T(k^n n^2) = \frac{kz^2+k^2z}{(z-k)^3} = \frac{kz(z+k)}{(z-k)^3}$$

Taking $k = e^{-a}$, we obtain

$$Z_T(e^{-an} n^2) = \frac{e^{-a} z(z+e^{-a})}{(z-e^{-a})^3}$$

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2. *Obtain the Z - transform of* $\cos n\theta$ *and* $\sin n\theta$. *Hence deduce Z-transforms of the following.*

(i) $k^n \cos n\theta$ **(ii)** $k^n \sin n\theta$ **(iii)** $e^{-an}\cos n\theta$ **(iv)** $e^{-an}\sin n\theta$

>> We know that $e^{in\theta} = \cos n\theta + i\sin n\theta$

We can write $e^{in\theta} = (e^{i\theta})^n = k^n$ where $k = e^{i\theta}$

We have $Z_T(k^n) = \dfrac{z}{z-k}$, k being $e^{i\theta}$

$$\therefore \quad Z_T(e^{in\theta}) = \frac{z}{z-e^{i\theta}} = \frac{z(z-e^{-i\theta})}{(z-e^{-i\theta})(z-e^{i\theta})}$$

$$= \frac{z[z-(\cos\theta - i\sin\theta)]}{z^2-z(e^{i\theta}+e^{-i\theta})+1}$$

$$= \frac{z[(z-\cos\theta)+i\sin\theta]}{z^2-2z\cos\theta+1}$$

ie., $$Z_T(\cos n\theta + i\sin n\theta) = \frac{z\left[(z-\cos\theta)+i\sin\theta\right]}{z^2-2z\cos\theta+1}$$

or $$Z_T(\cos n\theta) + i\,Z_T(\sin n\theta) = \left[\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}\right] + i\left[\frac{z\sin\theta}{z^2-2z\cos\theta+1}\right]$$

Equating the real and imaginary parts we get

$$Z_T(\cos n\theta) = \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$$

$$Z_T(\sin n\theta) = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

Now, suppose $Z_T(\cos n\theta) = \overline{u}(z)$ & $Z_T(\sin n\theta) = \overline{v}(z)$ then by damping rule,

$$Z_T(k^n \cos n\theta) = \overline{u}(z/k) \text{ and } Z_T(k^n \sin n\theta) = \overline{v}(z/k)$$

$$\therefore \quad Z_T(k^n \cos n\theta) = \frac{(z/k)(z/k - \cos\theta)}{(z/k)^2 - 2 \cdot z/k \cdot \cos\theta + 1}$$

Thus $$Z_T(k^n \cos n\theta) = \frac{z(z - k\cos\theta)}{z^2 - 2kz\cos\theta + k^2} \quad \ldots \text{(i)}$$

Also $$Z_T(k^n \sin n\theta) = \frac{z/k \cdot \sin\theta}{(z/k)^2 - 2 \cdot z/k \cdot \cos\theta + 1}$$

Thus $$Z_T(k^n \sin n\theta) = \frac{kz\sin\theta}{z^2 - 2kz\cos\theta + k^2} \quad \ldots \text{(ii)}$$

By taking $k = e^{-a}$ in (i) and (ii) we obtain the required results (iii) and (iv) as follows.

$$Z_T(e^{-an}\cos n\theta) = \frac{z(z - e^{-a}\cos\theta)}{z^2 - 2e^{-a}z\cos\theta + e^{-2a}} \quad \ldots \text{(iii)}$$

$$Z_T(e^{-an}\sin n\theta) = \frac{e^{-a}z\sin\theta}{z^2 - 2e^{-a}z\cos\theta + e^{-2a}} \quad \ldots \text{(iv)}$$

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3. *Find the Z - transform of* $(\cos\theta + i\sin\theta)^n$

>> We know that $\cos\theta + i\sin\theta = e^{i\theta}$

$$\therefore \quad (\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = k^n \text{ (say) where } k = e^{i\theta}$$

We have $Z_T(k^n) = \dfrac{z}{z-k}$

i.e., $Z_T(e^{in\theta}) = \dfrac{z}{z - e^{i\theta}}$

Thus $$Z_T\left[(\cos\theta + i\sin\theta)^n\right] = \frac{z}{z - e^{i\theta}}$$

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4. *Find the Z - transform of* $(n+1)^2$

>> $Z_T\left[(n+1)^2\right] = Z_T(n^2+2n+1)$

$$= Z_T(n^2) + 2\,Z_T(n) + Z_T(1)$$

$$= \frac{z^2+z}{(z-1)^3} + 2\cdot\frac{z}{(z-1)^2} + \frac{z}{z-1}$$

$$= \frac{z^2+z+2z(z-1)+z(z-1)^2}{(z-1)^3}$$

Thus $\quad \mathbf{Z_T\left[(n+1)^2\right] = \frac{z^3+z^2}{(z-1)^3}}$

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5. *Obtain the Z - transform of* $\cosh n\theta$ *and* $\sinh n\theta$.

>> $\cosh n\theta = \frac{1}{2}\left(e^{n\theta}+e^{-n\theta}\right) = \frac{1}{2}\left\{(e^{\theta})^n+(e^{-\theta})^n\right\}$

ie., $\quad \cosh n\theta = \frac{1}{2}\left\{p^n+q^n\right\}$ (*say*) where $p = e^{\theta}$ and $q = e^{-\theta}$

Now $Z_T(\cosh n\theta) = \frac{1}{2}\left\{Z_T(p^n)+Z_T(q^n)\right\}$

$$= \frac{1}{2}\left\{\frac{z}{z-p}+\frac{z}{z-q}\right\}$$

$$= \frac{z}{2}\left\{\frac{1}{z-e^{\theta}}+\frac{1}{z-e^{-\theta}}\right\}$$

$$= \frac{z}{2}\left\{\frac{z-e^{-\theta}+z-e^{\theta}}{(z-e^{\theta})(z-e^{-\theta})}\right\}$$

$$= \frac{z}{2}\cdot\left\{\frac{2z-(e^{\theta}+e^{-\theta})}{z^2-z(e^{\theta}+e^{-\theta})+1}\right\}$$

$$= \frac{z}{2}\left\{\frac{2z-2\cosh\theta}{z^2-2z\cosh\theta+1}\right\}$$

Thus $\quad \mathbf{Z_T(\cosh n\theta) = \frac{z(z-\cosh\theta)}{z^2-2z\cosh\theta+1}}$

Next, $\sinh n\,\theta = \dfrac{e^{n\theta} - e^{-n\theta}}{2}$

Proceeding on the same lines as before, we have,

$$Z_T(\sin h\, n\,\theta) = \frac{z}{2}\left\{\frac{1}{z - e^{\theta}} - \frac{1}{z - e^{-\theta}}\right\}$$

$$= \frac{z}{2}\left\{\frac{z - e^{-\theta} - z + e^{\theta}}{z^2 - 2z\cosh\theta + 1}\right\}$$

$$= \frac{z}{2} \cdot \frac{(e^{\theta} - e^{-\theta})}{z^2 - 2z\cosh\theta + 1}$$

$$= \frac{z}{2} \cdot \frac{2\sinh\theta}{z^2 - 2z\cosh\theta + 1}$$

Thus $$\boldsymbol{Z_T(\sin h\, n\,\theta) = \frac{z\sin h\theta}{z^2 - 2z\cos h\theta + 1}}$$

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6. *Obtain the Z-transform of* u_{n+1}, u_{u+2} *from the basic definition. Also give the Z-transform of* u_{n+k}

>> $$Z_T(u_{n+1}) = \sum_{n=0}^{\infty} u_{n+1} z^{-n} = z \sum_{0}^{\infty} \frac{1}{z} u_{n+1} z^{-n}$$

$$= z \sum_{0}^{\infty} u_{n+1} z^{-(n+1)}$$

$$= z\left\{u_1 z^{-1} + u_2 z^{-2} + \cdots\right\}$$

$$= z\left\{(u_0 + u_1 z^{-1} + u_2 z^{-2} + \cdots) - u_0\right\}$$

$$= z\left\{\sum_{n=0}^{\infty} u_n z^{-n} - u_0\right\}$$

Thus $$\boldsymbol{Z_T(u_{n+1}) = z\left[\bar{u}(z) - u_0\right]} \quad \ldots (1)$$

Next $$Z_T(u_{n+2}) = \sum_{n=0}^{\infty} u_{n+2}\, z^{-n} = z^2 \sum_{0}^{\infty} \frac{1}{z^2} u_{n+2} z^{-n}$$

$$Z_T(u_{n+2}) = z^2 \sum_0^\infty u_{n+2}\, z^{-(n+2)}$$

$$= z^2 \left\{ u_2 z^{-2} + u_3 z^{-3} + \cdots \right\}$$

$$= z^2 \left\{ (u_0 + u_1 z^{-1} + u_2 z^{-2} + u_3 z^{-3} + \cdots) - u_0 - u_1 z^{-1} \right\}$$

$$= z^2 \left\{ \sum_{n=0}^\infty u_n z^{-n} - u_0 - u_1 z^{-1} \right\}$$

Thus $\mathbf{Z_T(u_{n+2}) = z^2 \left[\bar{u}(z) - u_0 - u_1 z^{-1} \right]}$ . . . (2)

Similarly $Z_T(u_{n+3}) = z^3 \left[\bar{u}(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} \right]$ . . . (3)

In general we can write,

$$Z_T(u_{n+k}) = z^k \left[\bar{u}(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \cdots - u_{k-1} z^{-(k-1)} \right]$$

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7. *Starting from the definition of the Z-transform find the Z-transform of* **1**.

>> By the definition $Z_T(u_n) = \sum_{n=0}^\infty u_n z^{-n}$

$$\therefore \quad Z_T(1) = \sum_0^\infty z^{-n} = \sum_0^\infty \frac{1}{z^n}$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots \text{ is a geometric series.}$$

Hence $Z_T(1) = \dfrac{1}{1-(1/z)} = \dfrac{z}{z-1}$

Thus $\mathbf{Z_T(1) = \dfrac{z}{z-1}}$

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8. *Find the Z-transform of* $2n + \sin(n\pi/4) + 1$

>> Let $u_n = 2n + \sin(n\pi/4) + 1$

$$\therefore \quad Z_T(u_n) = 2Z_T(n) + Z_T\left[\sin(n\pi/4)\right] + Z_T(1)$$

i.e., $$Z_T(u_n) = \frac{2z}{(z-1)^2} + Z_T\left[\sin(n\pi/4)\right] + \frac{z}{z-1} \qquad \ldots(1)$$

We shall find $Z_T\left[\sin(n\pi/4)\right]$

We have $e^{in\pi/4} = \cos(n\pi/4) + i\sin(n\pi/4)$

But $e^{in\pi/4} = (e^{i\pi/4})^n = k^n$ *(say)* where $k = e^{i\pi/4}$

We know that $Z_T(k^n) = \dfrac{z}{z-k}$

$$\therefore \quad Z_T\left(e^{in\pi/4}\right) = \frac{z}{z - e^{i\pi/4}}$$

i.e., $$Z_T\left[\cos(n\pi/4) + i\sin(n\pi/4)\right] = \frac{z}{z - \left[\cos(\pi/4) + i\sin(\pi/4)\right]}$$

$$= \frac{z}{z - \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}}$$

$$Z_T\left[\cos(n\pi/4) + i\sin(n\pi/4)\right] = \frac{z\left[\left(z - \frac{1}{\sqrt{2}}\right) + \frac{i}{\sqrt{2}}\right]}{\left(z - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}}$$

$$\therefore \quad Z_T\left[\cos(n\pi/4) + i\sin(n\pi/4)\right] = \frac{z\left[\left(z - \frac{1}{\sqrt{2}}\right) + \frac{i}{\sqrt{2}}\right]}{z^2 - \sqrt{2}z + 1}$$

Equating the imaginary parts on both sides we get,

$$Z_T\left[\sin(n\pi/4)\right] = \frac{z/\sqrt{2}}{z^2 - \sqrt{2}z + 1} = \frac{z}{\sqrt{2}(z^2 - \sqrt{2}z + 1)}$$

We substitute this result in (1).

Thus $$\mathbf{Z_T(u_n) = \frac{2z}{(z-1)^2} + \frac{z}{\sqrt{2}(z^2 - \sqrt{2}z + 1)} + \frac{z}{z-1}}$$

---

9. *Using* $Z_T(n^2) = \dfrac{z^2+z}{(z-1)^3}$ *show that* $Z_T\left[(n+1)^2\right] = \dfrac{z^3+z^2}{(z-1)^3}$

>> Let $u_n = n^2$ and we have $Z_T(u_n) = Z_T(n^2) = \bar{u}(z)$

Consider the property $Z_T(u_{n+1}) = z\left[\bar{u}(z) - u_0\right]$ . . . (1)

Since $u_n = n^2$, $u_{n+1} = (n+1)^2$ and $u_0 = 0$. Hence (1) becomes

$$Z_T\left[(n+1)^2\right] = z\left[\frac{z^2+z}{(z-1)^3} - 0\right]$$

Thus $\mathbf{Z_T\left[(n+1)^2\right] = \dfrac{z^3+z^2}{(z-1)^3}}$

---

10. *Show that* $Z_T\left[\dfrac{1}{n!}\right] = e^{1/z}$ *Hence find* $Z_T\left[\dfrac{1}{(n+1)!}\right]$ *and* $Z_T\left[\dfrac{1}{(n+2)!}\right]$

>> By the definition,

$$Z_T\left[\frac{1}{n!}\right] = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = 1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \cdots$$

But $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$ and here we have $x = z^{-1}$

Thus $\mathbf{Z_T\left[\dfrac{1}{n!}\right] = e^{z^{-1}} = e^{1/z}}$

We have the properties,

$$Z_T(u_{n+1}) = z\left[\bar{u}(z) - u_0\right] \quad \ldots (1)$$

$$Z_T(u_{n+2}) = z\left[\bar{u}(z) - u_0 - u_1 z^{-1}\right] \quad \ldots (2)$$

Let $u_n = \dfrac{1}{n!}$ $\therefore$ $Z_T(u_n) = \bar{u}(z) = e^{1/z}$

Also $u_0 = \dfrac{1}{0!} = 1$ and $u_1 = \dfrac{1}{1!} = 1$

Thus by using these results in (1) and (2) we obtain,

$$\mathbf{Z_T\left[\frac{1}{(n+1)!}\right] = z\left[e^{1/z} - 1\right]}$$

$$\mathbf{Z_T\left[\frac{1}{(n+2)!}\right] = z\left[e^{1/z} - 1 - z^{-1}\right]}$$

---

11. *Find the Z - transform of* $\cos(n\pi/2 + \pi/4)$

>> Let $u_n = \cos(n\pi/2 + \pi/4)$

$= \cos(n\pi/2)\cos(\pi/4) - \sin(n\pi/2)\sin(\pi/4)$

i.e., $u_n = \frac{1}{\sqrt{2}}\left[\cos(n\pi/2) - \sin(n\pi/2)\right]$

$\therefore \quad Z_T(u_n) = \frac{1}{\sqrt{2}}\left[Z_T\cos(n\pi/2) - Z_T\sin(n\pi/2)\right] \qquad \ldots(1)$

Consider $e^{i(n\pi/2)} = \left(e^{i\pi/2}\right)^n = k^n$ (*say*) where $k = e^{i\pi/2}$

We know that $Z_T(k^n) = \frac{z}{z-k}$ and hence we have,

$$Z_T\left(e^{in\pi/2}\right) = \frac{z}{z - e^{i\pi/2}} = \frac{z}{z - \cos(\pi/2) - i\sin(\pi/2)} = \frac{z}{z-i}$$

$$Z_T\left(e^{in\pi/2}\right) = \frac{z(z+i)}{(z-i)(z+i)} = \frac{z^2 + iz}{z^2+1}$$

i.e., $Z_T\left[\cos(n\pi/2) + i\sin(n\pi/2)\right] = \frac{z^2}{z^2+1} + i\frac{z}{z^2+1}$

$\Rightarrow \quad Z_T[\cos(n\pi/2)] = \frac{z^2}{z^2+1}$ and $Z_T[\sin(n\pi/2)] = \frac{z}{z^2+1}$

We substitute these results in (1).

Thus $Z_T(u_n) = \frac{1}{\sqrt{2}}\left[\frac{z^2}{z^2+1} - \frac{z}{z^2+1}\right] = \frac{\mathbf{z(z-1)}}{\mathbf{\sqrt{2}(z^2+1)}}$

Note and Remember :

$$Z_T[\sin(n\pi/2)] = \frac{z}{z^2+1} \text{ and } Z_T[\cos(n\pi/2)] = \frac{z^2}{z^2+1}$$

---

12. *Find the Z - transform of* $\sin(3n+5)$

>> Let $u_n = \sin(3n+5) = \sin 3n\cos 5 + \cos 3n\sin 5$

$\therefore \quad Z_T(u_n) = \cos 5\, Z_T(\sin 3n) + \sin 5\, Z_T(\cos 3n) \qquad \ldots(1)$

Consider $e^{i(3n)} = (e^{3i})^n = k^n$ (*say*) where $k = e^{3i}$

We know that $Z_T(k^n) = \frac{z}{z-k}$

i.e., $$Z_T\left(e^{3in}\right) = \frac{z}{z-e^{3i}} = \frac{z}{(z-\cos 3) - i\sin 3}$$

$$= \frac{z\,[\,(z-\cos 3) + i\sin 3\,]}{(z-\cos 3)^2 + \sin^2 3}$$

i.e., $$Z_T(\cos 3n + i\sin 3n) = \frac{z(z-\cos 3)}{z^2 - 2z\cos 3 + 1} + i\,\frac{z\sin 3}{z^2 - 2z\cos 3 + 1}$$

$$\Rightarrow \quad Z_T(\cos 3n) = \frac{z(z-\cos 3)}{z^2 - 2z\cos 3 + 1} \text{ and } Z_T(\sin 3n) = \frac{z\sin 3}{z^2 - 2z\cos 3 + 1}$$

Substituting these results in (1) we get,

$$Z_T(u_n) = \cos 5 \cdot \frac{z\sin 3}{z^2 - 2z\cos 3 + 1} + \sin 5 \cdot \frac{z(z-\cos 3)}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z(\sin 3\cos 5 - \cos 3\sin 5) + z^2\sin 5}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z(-\sin 2) + z^2\sin 5}{z^2 - 2z\cos 3 + 1}$$

Thus $$\boldsymbol{Z_T(u_n) = \frac{z(z\sin 5 - \sin 2)}{z^2 - 2z\cos 3 + 1}}$$

---

13. *Find the Z -transform of* $\cos h\,(n\pi/2 + \theta)$

>> Let $u_n = \cos h\,(n\pi/2 + \theta) = \frac{1}{2}\left[e^{(n\pi/2+\theta)} + e^{-(n\pi/2+\theta)}\right]$

i.e., $$u_n = \frac{1}{2}\left[e^{\theta}e^{n\pi/2} + e^{-\theta}e^{-n\pi/2}\right]$$

$$\therefore \quad Z_T(u_n) = \frac{1}{2}\left[e^{\theta}Z_T(e^{n\pi/2}) + e^{-\theta}Z_T(e^{-n\pi/2})\right] \quad \ldots(1)$$

We have $Z_T(k^n) = \dfrac{z}{z-k}$

Taking $k = e^{\pi/2}$ and $e^{-\pi/2}$ we have

$$Z_T(e^{n\pi/2}) = \frac{z}{z-e^{\pi/2}} \text{ and } Z_T(e^{-n\pi/2}) = \frac{z}{z-e^{-\pi/2}}$$

Hence (1) becomes,

$$Z_T(u_n) = \frac{1}{2}\left[e^{\theta} \cdot \frac{z}{z - e^{\pi/2}} + e^{-\theta} \cdot \frac{z}{z - e^{-\pi/2}} \right]$$

$$= \frac{z}{2}\left[\frac{e^{\theta}(z - e^{-\pi/2}) + e^{-\theta}(z - e^{\pi/2})}{(z - e^{\pi/2})(z - e^{-\pi/2})} \right]$$

$$= \frac{z}{2}\left[\frac{z(e^{\theta} + e^{-\theta}) - \left\{ e^{(\pi/2-\theta)} + e^{-(\pi/2-\theta)} \right\}}{z^2 - z(e^{\pi/2} + e^{-\pi/2}) + 1} \right]$$

$$= \frac{z}{2}\left[\frac{2z\cos h\,\theta - 2\cos h\,(\pi/2 - \theta)}{z^2 - 2z\cos h\,(\pi/2) + 1} \right]$$

Thus $\quad \mathbf{Z_T(u_n) = \dfrac{z^2 \cos h\,\theta - z\cos h\,(\pi/2 - \theta)}{z^2 - 2z\cos h\,(\pi/2) + 1}}$

---

14. *Prove that* $Z_T(e^{-an} u_n) = \bar{u}(e^a z)$ *given that* $Z_T(u_n) = \bar{u}(z)$

>> By the definition

$$Z_T(e^{-an} u_n) = \sum_{n=0}^{\infty} e^{-an} u_n z^{-n} = \sum_{0}^{\infty} u_n (e^a)^{-n} z^{-n}$$

$$= \sum_{0}^{\infty} u_n (e^a z)^{-n} = \bar{u}(e^a z)$$

Thus $\quad \mathbf{Z_T(e^{-an} u_n) = \bar{u}(e^a z)}$

Note : 1. *This result can be deduced from the damping rule by taking* $k = e^a$

2. *This result is also referred to as the exponential shifting rule.*

---

15. *If* $u_n = (1/2)^n$, *show that* $Z_T(u_n) = \dfrac{2z}{2z-1}$ *from the definition.*

>> By the definition $Z_T\left[(1/2)^n\right] = \sum_{0}^{\infty} (1/2)^n z^{-n}$

i.e., $\quad Z_T\left[(1/2)^n\right] = \sum_{0}^{\infty} (2z)^{-n} = 1 + (2z)^{-1} + (2z)^{-2} + (2z)^{-3} + \cdots$

$$= 1 + (1/2z) + (1/2z)^2 + (1/2z)^3 + \cdots$$

The series in R.H.S being a geometric series we have,

$$Z_T\left[(1/2)^n\right] = \frac{1}{1-(1/2z)} = \frac{2z}{2z-1}$$

Thus $\mathbf{Z_T\,[\,(1/2)^n\,] = \dfrac{2z}{2z-1}}$

---

16. *Find the Z-transform of the unit step sequence* $u(n) = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$

\>> $$Z_T\,[\,u(n)\,] = \sum_{n=0}^{\infty} u(n)\,z^{-n}$$

$$= \sum_{n=0}^{\infty} 1 \cdot z^{-n}$$

$$= 1 + (1/z) + (1/z)^2 + (1/z)^3 + \cdots,$$ is a geometric series.

$\therefore$ $$Z_T\,[\,u(n)\,] = \frac{1}{1-(1/z)} = \frac{z}{z-1}$$

Thus $\mathbf{Z_T[\,u(n)\,] = \dfrac{z}{z-1}}$

---

17. *Find the Z-transform of the unit impulse sequence* $\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$

\>> $$Z_T[\,\delta(n)\,] = \sum_{n=0}^{\infty} \delta(n)\,z^{-n}$$

$$= \delta(0)\,z^0 + \delta(1)\,z^{-1} + \delta(2)\,z^{-2} + \cdots$$

$$= 1 + 0 + 0 + \cdots = 1$$

Thus $\mathbf{Z_T[\,\delta(n)\,] = 1}$

---

18. *If* $Z_T(u_n) = \bar{u}(z)$, *show that* $Z_T(n\,u_n) = -z\dfrac{d}{dz}[\,\bar{u}(z)\,]$

\>> $$Z_T(n\,u_n) = \sum_{n=0}^{\infty} (n\,u_n)\cdot z^{-n}$$

$$= -z\sum_{n=0}^{\infty} -n\,u_n\,z^{-n-1}$$

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$$Z_T(n\,u_n) = -z \sum_{n=0}^{\infty} u_n \frac{d}{dz}(z^{-n})$$

$$= -z \sum_{n=0}^{\infty} \frac{d}{dz}(u_n z^{-n})$$

$$= -z \frac{d}{dz} \sum_{n=0}^{\infty} u_n z^{-n} = -z \frac{d}{dz}[\bar{u}(z)]$$

Thus $\boldsymbol{Z_T(n\,u_n) = -z \frac{d}{dz}[\bar{u}(z)] = -z \frac{d}{dz}[Z_T(u_n)]}$

Remark : *In general we can show that*

$$Z_T[n^k u_n] = (-1)^k z^k \frac{d^k}{dz^k}[\bar{u}(z)]$$ *where k is a positive integer.*

---

19. *Find the Z-transform of n* cos *n* θ

>> Let $u_n = \cos n\theta$

We know that $Z_T(u_n) = Z_T(\cos\theta) = \dfrac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} = \bar{u}(z)$

We also have the property.

$$Z_T(n\,u_n) = -z\frac{d}{dz}[\bar{u}(z)]$$

Hence, $Z_T(n\cos n\theta) = -z\dfrac{d}{dz}\left[\dfrac{z^2-z\cos\theta}{z^2-2z\cos\theta+1}\right]$

ie., $$= -z\left\{\frac{(z^2-2z\cos\theta+1)(2z-\cos\theta)-(z^2-z\cos\theta)(2z-2\cos\theta)}{(z^2-2z\cos\theta+1)^2}\right\}$$

$$= \frac{-z}{(z^2-2z\cos\theta+1)^2}\left\{\begin{array}{c}2z^3-4z^2\cos^2\theta+2z-z^2\cos\theta+2z\cos^2\theta-\cos\theta\\ -(2z^3-2z^2\cos\theta-2z^2\cos\theta+2z\cos^2\theta)\end{array}\right\}$$

$$= \frac{-z}{(z^2-2z\cos\theta+1)^2}(2z-z^2\cos\theta-\cos\theta)$$

Thus $\boldsymbol{Z_T(n\cos n\theta) = \dfrac{z^3\cos\theta+z\cos\theta-2z^2}{(z^2-2z\cos\theta+1)^2}}$

---

20. *Find* $Z_T\left[\dfrac{1}{n+1}\right]$

>> $$Z_T\left[\frac{1}{n+1}\right] = \sum_{n=0}^{\infty} \frac{1}{n+1}\, z^{-n}$$

$$= 1 + \frac{1}{2} z^{-1} + \frac{1}{3} z^{-2} + \frac{1}{4} z^{-3} + \cdots$$

$$= 1 + \frac{1}{2z} + \frac{1}{3z^2} + \frac{1}{4z^3} + \cdots$$

or $$Z_T\left[\frac{1}{n+1}\right] = z\left[\frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \frac{1}{4z^4} + \cdots\right] \quad \ldots (1)$$

We have $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$

$\therefore$ $$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

or $$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

By taking $x = 1/z$ we have,

$$-\log\left(1 - \frac{1}{z}\right) = \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \frac{1}{4z^4} + \cdots$$

or $$-\log\left(\frac{z-1}{z}\right) = \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \frac{1}{4z^4} + \cdots$$

or $$\log\left(\frac{z}{z-1}\right) = \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \frac{1}{4z^4} + \cdots \quad \ldots (2)$$

We use (2) in the R.H.S of (1).

Thus $$\mathbf{Z_T\left[\frac{1}{n+1}\right] = z \log\left(\frac{z}{z-1}\right)}$$

---

21. *Find* $Z_T\left[\dfrac{1}{(n+1)(n+2)}\right]$

>> Let $u_n = \dfrac{1}{(n+1)(n+2)}$

$$\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2},$$ by partial fractions.

Now, $Z_T\,(u_n) = Z_T\left[\frac{1}{n+1}\right] - Z_T\left[\frac{1}{n+2}\right]$ . . . (1)

But, $Z_T\left[\frac{1}{n+1}\right] = z\log\left(\frac{z}{z-1}\right)$ . . . (2)

[*Refer Problem-20*]

Next, $Z_T\left[\frac{1}{n+2}\right] = \sum_{n=0}^{\infty} \frac{1}{n+2}\, z^{-n}$

$$= \frac{1}{2} + \frac{z^{-1}}{3} + \frac{z^{-2}}{4} + \cdots$$

ie., $Z_T\left[\frac{1}{n+2}\right] = \frac{1}{2} + \frac{x}{3} + \frac{x^2}{4} + \cdots$ where $x = z^{-1} = \frac{1}{z}$

$$= \frac{1}{x^2}\left(\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right)$$

$$= \frac{1}{x^2}\;-\log\,(1-x) - x$$ [*Refer Problem-20*]

$$= -z^2\,\log\,(1 - \frac{1}{z}) - z$$

$\therefore$ $Z_T\left[\frac{1}{n+2}\right] = -z^2\,\log\left(\frac{z-1}{z}\right) - z$ . . . (3)

Using (2) and (3) in (1) we have,

$$Z_T(u_n) = z\log\left(\frac{z}{z-1}\right) + z^2\log\left(\frac{z-1}{z}\right) + z$$

ie., $Z_T(u_n) = z\log\left(\frac{z}{z-1}\right) - z^2\log\left(\frac{z}{z-1}\right) + z$

Thus $\mathbf{Z_T\left[\frac{1}{(n+1)\,(n+2)}\right] = z\log\left(\frac{z}{z-1}\right)\{1-z\} + z}$

---

22. *Find (i)* $Z_T\left[{}^{(n+p)}C_p\right]$ *(ii)* $Z_T\left[a^n\,.\,{}^{(n+p)}C_p\right]$

>> (i) $Z_T\left[{}^{(n+p)}C_p\right] = \sum_{n=0}^{\infty} {}^{(n+p)}C_p\; z^{-n}$

$$= {}^{p}C_p + {}^{(1+p)}C_p\, z^{-1} + {}^{(2+p)}C_p\, z^{-2} + \cdots$$

Using the fundamental property of combinations,

$^{n}C_r = {}^{n}C_{n-r}$ and also $^{n}C_n = 1$ we have,

$$Z_T\left[{}^{(n+p)}C_p\right] = 1 + {}^{(1+p)}C_1\, z^{-1} + {}^{(2+p)}C_2\, z^{-2} + \cdots$$

$$Z_T\left[{}^{(n+p)}C_p\right] = 1 + (1+p)\, z^{-1} + \frac{(2+p)\,(1+p)}{2!}\, z^{-2} + \cdots \qquad \ldots(1)$$

We have $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}\, x^2 + \cdots$

$$\therefore \qquad (1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}\, x^2 + \cdots$$

Taking $x = z^{-1}$ and $n = 1+p$ in this expansion we have

$$(1-z^{-1})^{-(1+p)} = 1 + (1+p)\, z^{-1} + \frac{(1+p)\,(2+p)}{2!} \cdot z^{-2} + \cdots$$

Using this result in the R.H.S of (1) we get

$$Z_T\left[{}^{(n+p)}C_p\right] = (1-z^{-1})^{-(1+p)} = \left(\frac{z-1}{z}\right)^{-(1+p)}$$

Thus $\quad \mathbf{Z_T\left[{}^{(n+p)}C_p\right] = \left(\dfrac{z}{z-1}\right)^{1+p}}$

(ii) We have the damping rule,

$$Z_T(a^n \cdot u_n) = \bar{u}(z/a) \quad \text{where} \quad Z_T(u_n) = \bar{u}(z)$$

Now $\quad Z_T\left[a^n \cdot {}^{n+p}C_p\right] = \left[\dfrac{z/a}{(z/a)-1}\right]^{1+p}$

Thus $\quad \mathbf{Z_T\left[a^n \cdot {}^{(n+p)}C_p\right] = \left[\dfrac{z}{z-a}\right]^{1+p}}$

8.7 Initial Value Theorem

Statement : *If* $Z_T(u_n) = \bar{u}(z)$ *then* $\lim\limits_{z \to \infty} \bar{u}(z) = u_0$

Proof : We have by the definition,

$$Z_T(u_n) = \sum_{n=0}^{\infty} u_n\, z^{-n}$$

ie., $\qquad \bar{u}(z) = u_0 + u_1 z^{-1} + u_2 z^{-2} + u_3 z^{-3} + \cdots \qquad \ldots(1)$

$$\therefore \quad \underset{z \to \infty}{lt}\ \bar{u}(z) = \underset{z \to \infty}{lt}\left[u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \cdots\right] = u_0 + 0 + 0 + \cdots = u_0$$

Thus $\underset{z \to \infty}{lt}\ \bar{u}(z) = u_0$

Remark : *Similarly we can also obtain other initial values* $u_1,\ u_2, \cdots$ *as follows.*

We have from (1),

$$\bar{u}(z) - u_0 = \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \cdots$$

or $$z[\bar{u}(z) - u_0] = u_1 + \frac{u_2}{z} + \frac{u_3}{z^2} + \cdots$$

$$\therefore \quad \underset{z \to \infty}{lt}\ z[\bar{u}(z) - u_0] = u_1$$

Also $$\bar{u}(z) - u_0 - \frac{u_1}{z} = \frac{u_2}{z^2} + \frac{u_3}{z^3} + \cdots$$

$$\therefore \quad \underset{z \to \infty}{lt}\ z^2\left\{\bar{u}(z) - u_0 - \frac{u_1}{z}\right\} = u_2$$

8.8 Final Value Theorem

Statement : *If* $Z_T(u_n) = \bar{u}(z)$ *then,* $\underset{z \to 1}{lt}\left[(z-1)\,\bar{u}(z)\right] = \underset{n \to \infty}{lt}\ u_n$

Proof : We have the result,

$$Z_T(u_{n+1}) = z[\bar{u}(z) - u_0] \quad \ldots (1)$$

Also, $$Z_T(u_n) = \bar{u}(z) \quad \ldots (2)$$

Now (1) - (2) will give us,

$$Z_T(u_{n+1}) - Z_T(u_n) = z\,\bar{u}(z) - z\,u_0 - \bar{u}(z)$$

or $$Z_T(u_{n+1} - u_n) = (z-1)\,\bar{u}(z) - z\,u_0$$

ie., $$\sum_{n=0}^{\infty}(u_{n+1} - u_n)\,z^{-n} = (z-1)\,\bar{u}(z) - z\,u_0$$

Now taking limit as $z \to 1$ we have,

$$\lim_{z \to 1} \sum_{n=0}^{\infty} (u_{n+1} - u_n) \frac{1}{z^n} = \lim_{z \to 1} \left[(z-1)\, \overline{u}(z) - z\, u_0 \right]$$

ie., $$\sum_{n=0}^{\infty} (u_{n+1} - u_n) = \lim_{z \to 1} \left[(z-1)\, \overline{u}(z) \right] - u_0$$

or $$\lim_{z \to 1} \left[(z-1)\, \overline{u}(z) \right] = u_0 + \sum_{n=0}^{\infty} (u_{n+1} - u_n)$$

$$= u_0 + \lim_{m \to \infty} \sum_{n=0}^{m} (u_{n+1} - u_n)$$

$$= u_0 + \lim_{m \to \infty} \left\{ (u_1 - u_0) + (u_2 - u_1) + (u_3 - u_2) + \cdots + (u_m - u_{m-1}) + (u_{m+1} - u_m) \right\}$$

$$= u_0 + \lim_{m \to \infty} (-u_0 + u_{m+1})$$

$$= \lim_{m \to \infty} u_{m+1}$$

It should be noted that as $m \to \infty$, $(m+1)$ also tends to ∞ and hence R.H.S is equivalent to $\lim_{(m+1) \to \infty} u_{m+1}$ which is further equivalent to $\lim_{n \to \infty} u_n$

Thus we have proved that,

$$\lim_{z \to 1} \left[(z-1)\, \overline{u}(z) \right] = \lim_{n \to \infty} u_n$$

WORKED PROBLEMS

23. *If* $\overline{u}(z) = \dfrac{2z^2 + 3z + 12}{(z-1)^4}$, *find the value of* u_0, u_1, u_2, u_3

>> We have by the definition of Z-transform,

$$\overline{u}(z) = u_0 + u_1 z^{-1} + u_2 z^{-2} + u_3 z^{-3} + u_4 z^{-4} + \cdots$$

ie., $$\overline{u}(z) = u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \frac{u_4}{z^4} + \cdots$$

Therefore we have,

$$u_0 = \lim_{z \to \infty} \overline{u}(z) \qquad \ldots (1)$$

$$u_1 = \underset{z \to \infty}{lt} \; z\left[\bar{u}(z) - u_0\right] \qquad \ldots (2)$$

$$u_2 = \underset{z \to \infty}{lt} \; z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right] \qquad \ldots (3)$$

$$u_3 = \underset{z \to \infty}{lt} \; z^3\left[\bar{u}(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2}\right] \qquad \ldots (4)$$

From (1), $u_0 = \underset{z \to \infty}{lt} \; \dfrac{2z^2 + 3z + 12}{(z-1)^4}$

$$= \underset{z \to \infty}{lt} \; \frac{z^2(2 + 3/z + 12/z^2)}{z^4(1 - 1/z)^4}$$

$$= \underset{z \to \infty}{lt} \; \frac{1}{z^2} \cdot \frac{(2 + 3/z + 12/z^2)}{(1 - 1/z)^4} = 0 \cdot \frac{2}{1} = 0$$

$\therefore \qquad$ **$u_0 = 0$**

From (2), $u_1 = \underset{z \to \infty}{lt} \; z \cdot \dfrac{2z^2 + 3z + 12}{(z-1)^4}$, since $u_0 = 0$

$$= \underset{z \to \infty}{lt} \; z^3 \cdot \frac{(2 + 3/z + 12/z^2)}{z^4(1 - 1/z)^4}$$

$$= \underset{z \to \infty}{lt} \; \frac{1}{z} \cdot \frac{(2 + 3/z + 12/z^2)}{(1 - 1/z)^4} = 0$$

$\therefore \qquad$ **$u_1 = 0$**

From (3), $u_2 = \underset{z \to \infty}{lt} \; z^2\left[\dfrac{2z^2 + 3z + 12}{(z-1)^4}\right]$ since $u_0 = 0 = u_1$

$$= \underset{z \to \infty}{lt} \; z^4 \cdot \frac{(2 + 3/z + 12/z^2)}{z^4(1 - 1/z)^4} = \frac{2 + 0 + 0}{(1-0)^4} = 2$$

$\therefore \qquad$ **$u_2 = 2$**

From (4), $u_3 = \underset{z \to \infty}{lt} \; z^3\left[\dfrac{2z^2 + 3z + 12}{(z-1)^4} - \dfrac{2}{z^2}\right]$

(We need to simplify here)

$$u_3 = \underset{z \to \infty}{lt} \; z^3 \left[\frac{2z^4 + 3z^3 + 12z^2 - 2(z^4 - 4z^3 + 6z^2 - 4z + 1)}{z^2 (z-1)^4} \right]$$

$$= \underset{z \to \infty}{lt} \; z \left[\frac{11z^3 + 8z - 2}{(z-1)^4} \right]$$

$$= \underset{z \to \infty}{lt} \; z^4 \cdot \frac{(11 + 8/z^2 - 2/z^3)}{z^4 (1 - 1/z)^4} = 11$$

$\therefore$ $\quad \boldsymbol{u_3 = 11}$

Thus $\quad \boldsymbol{u_0 = 0,\ u_1 = 0,\ u_2 = 2}$ **and** $\boldsymbol{u_3 = 11}$

---

24. *Given* $Z_T(u_n) = \dfrac{2z^2 + 3z + 4}{(z-3)^3}$, $|z| > 3$ *show that* $u_1 = 2$, $u_2 = 21$, $u_3 = 139$

>> [*As in Problem-23, results (1) to (4) need to be given*]

$$u_0 = \underset{z \to \infty}{lt} \; \frac{2z^2 + 3z + 4}{(z-3)^3}$$

$$= \underset{z \to \infty}{lt} \; \frac{z^2 (2 + 3/z + 4/z^2)}{z^3 (1 - 3/z)^3}$$

$$= \underset{z \to \infty}{lt} \; \frac{1}{z} \cdot \frac{(2 + 3/z + 4/z^2)}{(1 - 3/z)^3} = 0$$

$\therefore$ $\quad u_0 = 0$

$$u_1 = \underset{z \to \infty}{lt} \; z \left[\frac{2z^2 + 3z + 4}{(z-3)^3} \right] \quad \text{since } u_0 = 0$$

$$= \underset{z \to \infty}{lt} \; z^3 \cdot \frac{(2 + 3/z + 4/z^2)}{z^3 (1 - 3/z)^3} = 2$$

$\therefore$ $\quad \boldsymbol{u_1 = 2}$

$$u_2 = \underset{z \to \infty}{lt} \; z^2 \left[\frac{2z^2 + 3z + 4}{(z-3)^3} - 0 - \frac{2}{z} \right]$$

$$= \underset{z \to \infty}{lt} \; z^2 \left[\frac{2z^3 + 3z^2 + 4z - 2(z^3 - 9z^2 + 27z - 27)}{z(z-3)^3} \right]$$

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$$u_2 = \lim_{z \to \infty} \frac{z(21z^2 - 50z + 54)}{(z-3)^3}$$

$$= \lim_{z \to \infty} \frac{z^3(21 - 50/z + 54/z^2)}{z^3(1 - 3/z)^3} = 21$$

$\therefore \quad \mathbf{u_2 = 21}$

$$u_3 = \lim_{z \to \infty} z^3 \left[\frac{2z^2 + 3z + 4}{(z-3)^3} - \frac{2}{z} - \frac{21}{z^2} \right]$$

$$= \lim_{z \to \infty} z^3 \left[\frac{z^3(2z^2 + 3z + 4) - 2z^2(z-3)^3 - 21z(z-3)^3}{z^3(z-3)^3} \right]$$

$$= \lim_{z \to \infty} \left[\frac{(2z^5 + 3z^4 + 4z^3) - (2z^2 + 21z)(z^3 - 9z^2 + 27z - 27)}{(z-3)^3} \right]$$

$$= \lim_{z \to \infty} \left[\frac{(2z^5 + 3z^4 + 4z^3) - (2z^5 + 3z^4 - 135z^3 + 513z^2 - 567z)}{(z-3)^3} \right]$$

$$= \lim_{z \to \infty} \left[\frac{139z^3 - 513z^2 + 567z}{(z-3)^3} \right]$$

$$= \lim_{z \to \infty} \frac{z^3(139 - 513/z + 567/z^2)}{z^3(1 - 3/z)^3} = 139$$

$\therefore \quad \mathbf{u_3 = 139}$

Thus we have proved that $\mathbf{u_1 = 2,\ u_2 = 21 \text{ and } u_3 = 139}$

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25. *Given* $Z_T(u_n) = \dfrac{z}{z-1} + \dfrac{z}{z^2+1}$ *obtain the Z-transform of* u_{n+2}

>> We have $Z_T(u_{n+2}) = z^2 \left[\bar{u}(z) - u_0 - \dfrac{u_1}{z} \right]$ . . . (1)

We shall first compute u_0 and u_1.

$$u_0 = \lim_{z \to \infty} \bar{u}(z)$$

$$= \lim_{z \to \infty} \left[\frac{z}{z-1} + \frac{z}{z^2+1} \right]$$

$$= \lim_{z \to \infty} \left[\frac{z}{z(1-1/z)} + \frac{z}{z^2(1+1/z^2)} \right]$$

$$= \lim_{z \to \infty} \left[\frac{1}{1-1/z} + \frac{1}{z} \cdot \frac{1}{(1+1/z^2)} \right] = 1 + 0 = 1$$

$\therefore \qquad u_0 = 1$

$$u_1 = \lim_{z \to \infty} z\,[\bar{u}(z) - u_0]$$

$$= \lim_{z \to \infty} z \left[\frac{z}{z-1} + \frac{z}{z^2+1} - 1 \right]$$

$$= \lim_{z \to \infty} z \left[\frac{z^3 + z + z^2 - z - (z^3 - z^2 + z - 1)}{(z-1)(z^2+1)} \right]$$

$$= \lim_{z \to \infty} z \left[\frac{2z^2 - z + 1}{(z-1)(z^2+1)} \right]$$

$$= \lim_{z \to \infty} \frac{z^3(2 - 1/z + 1/z^2)}{z(1-1/z)\, z^2(1+1/z^2)} = 2$$

$\therefore \qquad u_1 = 2$

Now from (1) we have,

$$Z_T(u_{n+2}) = z^2 \left[\frac{z}{z-1} + \frac{z}{z^2+1} - 1 - \frac{2}{z} \right]$$

$$= z^2 \left[\frac{z^3 + z^2}{(z-1)(z^2+1)} - \frac{(z+2)}{z} \right]$$

$$= \frac{z^2(z^2 - z + 2)}{(z-1)(z^2+1)(z)}, \quad \text{on simplification.}$$

Thus $Z_T(u_{n+2}) = \dfrac{z(z^2 - z + 2)}{(z-1)(z^2+1)}$

EXERCISES

1. Show that the Z - transform of n^4 is $\dfrac{z^4 + 11z^3 + 11z^2 + z}{(z-1)^5}$

2. Find the Z - transform of

   (i) e^{an} (ii) $e^{an} \cdot n$ (iii) $e^{an} \cdot n^2$

3. Obtain the Z - transform of

   (i) $(n+2)^2$ (ii) $(n+1)^3$ (iii) k^{n+4}

4. Show that the Z-transform of $\dfrac{a^n}{n!} e^{-a}$ is $e^{a/z}$

5. Show that

   (i) $Z_T(a^n \cosh n\theta) = \dfrac{z(z - a\cosh\theta)}{z^2 - 2az\cosh\theta + a^2}$

   (ii) $Z_T(a^n \sinh n\theta) = \dfrac{az\sinh\theta}{z^2 - 2az\cosh\theta + a^2}$

6. Show that the Z-transform of $\cos(n\pi/8 + \theta)$ is
   $\dfrac{z^2\cos\theta - z\cos(\pi/8 - \theta)}{z^2 - 2z\cos(\pi/8) + 1}$

7. Starting from the basic definition of Z-transform show that
   $Z_T[(1/3)^n] = \dfrac{3z}{3z-1}$

8. Show that $Z_T[\sin(n+1)\theta] = \dfrac{z^2\sin\theta}{z^2 - 2z\cos\theta + 1}$

9. If $\bar{u}(z) = \dfrac{2z^2 + 5z + 14}{(z-1)^4}$ show that $u_0 = 0 = u_1$, $u_2 = 2$ and $u_3 = 13$

10. If $Z_T(u_n) = \dfrac{5z^2 + 3z + 12}{(z-1)^4}$ show that $u_2 = 5$ and $u_3 = 23$

ANSWERS

2. (i) $\dfrac{z}{z-e^{a}}$ (ii) $\dfrac{z\,e^{a}}{(z-e^{a})^2}$ (iii) $\dfrac{e^{a}(z+e^{a})}{(z-e^{a})^3}$

3. (i) $\dfrac{z^2+z}{(z-1)^3}+\dfrac{4z}{(z-1)^2}+\dfrac{4z}{z-1}$

(ii) $\dfrac{z^3+4z^2+z}{(z-1)^4}+\dfrac{3(z^2+z)}{(z-1)^2}+\dfrac{3z}{(z-1)^2}+\dfrac{z}{z-1}$

(iii) $\dfrac{k^4 z}{z-k}$

8.9 Inverse Z-Transforms

We have already stated that, if $Z_T(u_n) = \bar{u}(z)$ then $Z_T^{-1}[\bar{u}(z)] = u_n$ is called the inverse Z-transform of $\bar{u}(z)$

List of standard inverse Z-transforms

1. $Z_T^{-1}\left[\dfrac{z}{z-1}\right] = 1$	2. $Z_T^{-1}\left[\dfrac{z}{z-k}\right] = k^n$
3. $Z_T^{-1}\left[\dfrac{z}{(z-1)^2}\right] = n$	4. $Z_T^{-1}\left[\dfrac{kz}{(z-k)^2}\right] = k^n \cdot n$
5. $Z_T^{-1}\left[\dfrac{z^2+z}{(z-1)^3}\right] = n^2$	6. $Z_T^{-1}\left[\dfrac{kz^2+k^2z}{(z-k)^3}\right] = k^n \cdot n^2$
7. $Z_T^{-1}\left[\dfrac{z^3+4z^2+z}{(z-1)^4}\right] = n^3$	8. $Z_T^{-1}\left[\dfrac{kz^3+4k^2z^2+k^3z}{(z-k)^4}\right] = k^n \cdot n^3$
9. $Z_T^{-1}\left[\dfrac{z}{z^2+1}\right] = \sin(n\pi/2)$	10. $Z_T^{-1}\left[\dfrac{z^2}{z^2+1}\right] = \cos(n\pi/2)$

Type-1.

Inverse Z-transform of rational algebraic functions by partial fractions method.

Working procedure for problems

➲ Given $\bar{u}(z) = \dfrac{f(z)}{g(z)}$ we need to express $g(z)$ in terms of non repeated linear factors only.

- We consider $\frac{\bar{u}(z)}{z}$ in the form of a proper fraction and resolve into partial fractions.
- We multiply by z to have $\bar{u}(z)$ involving various terms of the form $c \cdot (z/z-k)$, c being a constant.
- Finally we compute the inverse Z-transform of these terms resulting in the required $Z_T^{-1}[\bar{u}(z)]$

Important Note : *If $g(z)$ involves repeated linear factors of the form : $(z-k)^2$, $(z-k)^3$, $(z-k)^4 \cdots$ we need to take it into account the corresponding terms in the numerator : kz, (kz^2+k^2z), $(kz^3+4k^2z^2+k^3z)$, $\cdots$ (by refering into inverse Z-transform) respectively and express $\bar{u}(z)$ suitably with terms multiplied by $A, B, C, \cdots$*

We compute $A, B, C, \cdots$ and find $Z_T^{-1}[\bar{u}(z)]$

WORKED PROBLEMS

26. *Find the inverse Z-transform of* $\frac{z}{(z-1)(z-2)}$

>> Let $\bar{u}(z) = \frac{z}{(z-1)(z-2)}$

$\therefore \quad \frac{\bar{u}(z)}{z} = \frac{1}{(z-1)(z-2)}$

Let $\quad \frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$

or $\quad 1 = A(z-2) + B(z-1)$

Put $\quad z = 1 : 1 = A(-1) \qquad \therefore \quad A = -1$

Put $\quad z = 2 : 1 = B(1) \qquad \therefore \quad B = 1$

Hence $\quad \frac{\bar{u}(z)}{z} = \frac{-1}{z-1} + \frac{1}{z-2}$

or $\quad \bar{u}(z) = -\frac{z}{z-1} + \frac{z}{z-2}$

$\Rightarrow \quad Z_T^{-1}[\bar{u}(z)] = Z_T^{-1}\left[\frac{z}{z-2}\right] - Z_T^{-1}\left[\frac{z}{z-1}\right]$

We have $Z_T^{-1}\left[\frac{z}{z-k}\right] = k^n$

Thus $\quad \mathbf{Z_T^{-1}[\bar{u}(z)] = u_n = 2^n - 1}$

---

27. *Find* $Z_T^{-1}\left[\dfrac{5z}{(2-z)(3z-1)}\right]$

>> Let $\bar{u}(z) = \dfrac{5z}{(2-z)(3z-1)}$

$$\therefore \quad \frac{\bar{u}(z)}{z} = \frac{5}{(2-z)(3z-1)}$$

Let $\dfrac{5}{(2-z)(3z-1)} = \dfrac{A}{2-z} + \dfrac{B}{3z-1}$

or $5 = A(3z-1) + B(2-z)$

Put $z = 2 : 5 = A(5)$ $\therefore A = 1$

Put $z = 1/3 : 5 = B(2-1/3)$ or $5 = B(5/3)$ $\therefore B = 3$

Hence $\dfrac{\bar{u}(z)}{z} = \dfrac{1}{2-z} + \dfrac{3}{3z-1}$

or $\bar{u}(z) = \dfrac{-z}{z-2} + \dfrac{3z}{3(z-1/3)}$

$$\Rightarrow \quad Z_T^{-1}[\bar{u}(z)] = -Z_T^{-1}\left[\frac{z}{z-2}\right] + Z_T^{-1}\left[\frac{z}{z-1/3}\right]$$

Thus $\boldsymbol{Z_T^{-1}[\bar{u}(z)] = u_n = -2^n + (1/3)^n}$

---

28. *Compute the inverse Z-transform of* $\dfrac{3z^2+2z}{(5z-1)(5z+2)}$

Let $\bar{u}(z) = \dfrac{3z^2+2z}{(5z-1)(5z+2)}$

$$\therefore \quad \frac{\bar{u}(z)}{z} = \frac{3z+2}{(5z-1)(5z+2)}$$

Let $\dfrac{3z+2}{(5z-1)(5z+2)} = \dfrac{A}{5z-1} + \dfrac{B}{5z+2}$

or $3z+2 = A(5z+2) + B(5z-1)$

Put $z = 1/5 : 13/5 = A(3)$ $\therefore A = 13/15$

Put $z = -2/5 : 4/5 = B(-3)$ $\therefore B = -4/15$

Hence $\dfrac{\bar{u}(z)}{z} = \dfrac{13}{15}\dfrac{1}{5z-1} - \dfrac{4}{15}\dfrac{1}{5z+2}$

or $\quad \bar{u}(z) = \frac{13}{75} \frac{z}{z-(1/5)} - \frac{4}{75} \frac{z}{z+(2/5)}$

$\Rightarrow \quad Z_T^{-1}[\bar{u}(z)] = \frac{13}{75} Z_T^{-1}\left[\frac{z}{z-(1/5)}\right] - \frac{4}{75} Z_T^{-1}\left[\frac{z}{z+(2/5)}\right]$

Thus $\quad \mathbf{Z_T^{-1}[\bar{u}(z)] = u_n = \frac{1}{75}\left\{13(1/5)^n - 4(-2/5)^n\right\}}$ ■

---

29. *Obtain the inverse Z transform of* $\frac{2z^2+3z}{(z+2)(z-4)}$

>> Let $\bar{u}(z) = \frac{2z^2+3z}{(z+2)(z-4)}$

$\therefore \quad \frac{\bar{u}(z)}{z} = \frac{2z+3}{(z+2)(z-4)}$

Let $\quad \frac{2z+3}{(z+2)(z-4)} = \frac{A}{z+2} + \frac{B}{z-4}$

or $\quad 2z+3 = A(z-4) + B(z+2)$

Put $\quad z = -2 : -1 = A(-6) \quad \therefore A = 1/6$

Put $\quad z = 4 : 11 = B(6) \quad \therefore B = 11/6$

Hence $\frac{\bar{u}(z)}{z} = \frac{1}{6} \cdot \frac{1}{z+2} + \frac{11}{6} \cdot \frac{1}{z-4}$

or $\quad \bar{u}(z) = \frac{1}{6} \cdot \frac{z}{z+2} + \frac{11}{6} \cdot \frac{z}{z-4}$

$\Rightarrow \quad Z_T^{-1}[\bar{u}(z)] = \frac{1}{6} Z_T^{-1}\left[\frac{z}{z+2}\right] + \frac{11}{6} Z_T^{-1}\left[\frac{z}{z-4}\right]$

Thus $\quad \mathbf{Z_T^{-1}\left[\bar{u}(z)\right] = u_n = \frac{1}{6}\left\{(-2)^n + 11(4)^n\right\}}$

---

30. *Find* $Z_T^{-1}\left[\frac{8z-z^3}{(4-z)^3}\right]$

>> [*We have to note that the denominator has repeated linear factors*]

Let $\quad \bar{u}(z) = \frac{8z-z^3}{(4-z)^3} = \frac{z^3-8z}{(z-4)^3}$

We have $Z_T^{-1}\left[\dfrac{z}{z-4}\right] = 4^n$, $Z_T^{-1}\left[\dfrac{4z}{(z-4)^2}\right] = 4^n \cdot n$

and $\quad Z_T^{-1}\left[\dfrac{4z^2+16z}{(z-4)^3}\right] = 4^n \cdot n^2$

We resolve $\bar{u}(z)$ as follows. (*Note this important step*)

Let $\quad \dfrac{z^3-8z}{(z-4)^3} = A \cdot \dfrac{z}{z-4} + B \cdot \dfrac{4z}{(z-4)^2} + C \cdot \dfrac{4z^2+16z}{(z-4)^3}$ . . . (1)

or $\quad \dfrac{z^3-8z}{(z-4)^3} = \dfrac{Az(z-4)^2 + 4Bz(z-4) + 4Cz(z+4)}{(z-4)^3}$

or $\quad z^2 - 8 = A(z-4)^2 + 4B(z-4) + 4C(z+4)$

Put $\quad z = 4 : \; 8 = 4C(8) \qquad \therefore \quad C = 1/4$

Equating the coefficient of z^2 on both sides we have, $A = 1$

Also by equating the coefficient of z on both sides we have,

$$-8A + 4B + 4C = 0$$

ie., $\quad -8 + 4B + 1 = 0 \qquad \therefore \quad B = 7/4$

Substituting the values of A , B , C in (1) we have,

$$\frac{z^3-8z}{(z-4)^3} = \frac{z}{z-4} + \frac{7}{4} \cdot \frac{4z}{(z-4)^2} + \frac{1}{4} \cdot \frac{4z^2+16z}{(z-4)^3}$$

$$\Rightarrow \quad Z_T^{-1}\left[\frac{z^3-8z}{(z-4)^3}\right] = Z_T^{-1}\left[\frac{z}{z-4}\right] + \frac{7}{4} Z_T^{-1}\left[\frac{4z}{(z-4)^2}\right] + \frac{1}{4} Z_T^{-1}\left[\frac{4z^2+16z}{(z-4)^3}\right]$$

That is, $Z_T^{-1}[\bar{u}(z)] = u_n = 4^n + \dfrac{7}{4} 4^n \cdot n + \dfrac{1}{4} 4^n \cdot n^2$

Thus $\mathbf{Z_T^{-1}[\bar{u}(z)] = u_n = \dfrac{4^n}{4}(4 + 7n + n^2) = 4^{n-1}(4 + 7n + n^2)}$

---

31. *Given* $U(z) = \dfrac{4z^2-2z}{z^3-5z^2+8z-4}$, *find* u_n

>> $U(z)$ or $\bar{u}(z) = \dfrac{4z^2-2z}{z^3-5z^2+8z-4}$ by data.

We shall factorize the denominator first.

$$z^3 - 5z^2 + 8z - 4 = (z^3 - 5z^2 + 4z) + (4z - 4)$$
$$= z(z^2 - 5z + 4) + 4(z - 1)$$
$$= z(z-1)(z-4) + 4(z-1)$$
$$= (z-1)(z^2 - 4z + 4)$$
$$= (z-1)(z-2)^2$$

We have $\bar{u}(z) = \dfrac{4z^2 - 2z}{(z-1)(z-2)^2}$

We have $Z_T^{-1}\left[\dfrac{z}{z-1}\right] = 1,\ Z_T^{-1}\left[\dfrac{z}{z-2}\right] = 2^n,\ Z_T^{-1}\left[\dfrac{2z}{(z-2)^2}\right] = 2^n \cdot n$

We resove $\bar{u}(z)$ as follows.

$$\bar{u}(z) = \frac{4z^2 - 2z}{(z-1)(z-2)^2} = A \cdot \frac{z}{z-1} + B \cdot \frac{z}{z-2} + C \cdot \frac{2z}{(z-2)^2} \qquad \ldots (1)$$

or $$\frac{4z^2 - 2z}{(z-1)(z-2)^2} = \frac{Az(z-2)^2 + Bz(z-1)(z-2) + 2Cz(z-1)}{(z-1)(z-2)^2}$$

or $4z - 2 = A(z-2)^2 + B(z-1)(z-2) + 2C(z-1)$

Put $z = 1 : 2 = A(1) \quad \therefore \quad A = 2$

Put $z = 2 : 6 = 2C(1) \quad \therefore \quad C = 3$

Equating the coefficient of z^2 on both sides we have,

$$A + B = 0 \qquad \therefore \qquad B = -2$$

Substituting the values of A, B, C in (1) and taking inverse we have

$$Z_T^{-1}[\bar{u}(z)] = 2Z_T^{-1}\left[\frac{z}{z-1}\right] - 2Z_T^{-1}\left[\frac{z}{z-2}\right] + 3Z_T^{-1}\left[\frac{2z}{(z-2)^2}\right]$$
$$= 2.1 - 2.2^n + 3 \cdot 2^n \cdot n$$

Thus $\mathbf{Z_T^{-1}[\bar{u}(z)] = u_n = 2 - 2^{n+1} + 3n \cdot 2^n}$

---

32. *Find the inverse Z-transform of* $\dfrac{z^3 - 20z}{(z-2)^3(z-4)}$

>> Let $\bar{u}(z) = \dfrac{z^3 - 20z}{(z-2)^3(z-4)}$

We have

$$Z_T^{-1}\left[\frac{z}{z-2}\right] = 2^n, \qquad Z_T^{-1}\left[\frac{2z}{(z-2)^2}\right] = 2^n\, n$$

$$Z_T^{-1}\left[\frac{2z^2+4z}{(z-2)^3}\right] = 2^n \cdot n^2, \quad Z_T^{-1}\left[\frac{z}{z-4}\right] = 4^n$$

We resolve $\bar{u}(z)$ as follows.

$$\bar{u}(z) = \frac{z^3 - 20z}{(z-2)^3(z-4)} = A\cdot\frac{z}{z-2} + B\cdot\frac{2z}{(z-2)^2} + C\cdot\frac{2z^2+4z}{(z-2)^3} + D\cdot\frac{z}{z-4} \quad \ldots(1)$$

$$\frac{z^3 - 20z}{(z-2)^3(z-4)} = \frac{Az(z-2)^2(z-4) + 2Bz(z-2)(z-4) + C(2z^2+4z)(z-4) + Dz(z-2)^3}{(z-2)^3(z-4)}$$

or $z^2 - 20 = A(z-2)^2(z-4) + 2B(z-2)(z-4) + C(2z+4)(z-4) + D(z-2)^3$

Put $z = 2$: $-16 = -16C$ $\quad\therefore\ C = 1$

Put $z = 4$: $-4 = D(8)$ $\quad\therefore\ D = -1/2$

Equating the coefficient of z^3 on both sides we have,

$A + D = 0$ $\quad\therefore\ A = 1/2$

Put $z = 0$: $-20 = A(4)(-4) + 2B(8) + C(-16) + D(-8)$

ie., $-20 = -8 + 16B - 16 + 4$ $\quad\therefore\ B = 0$

Substituting the values of A, B, C, D in (1) and taking inverse we have,

$$Z_T^{-1}[\bar{u}(z)] = \frac{1}{2}Z_T^{-1}\left[\frac{z}{z-2}\right] + Z_T^{-1}\left[\frac{2z^2+4z}{(z-2)^3}\right] - \frac{1}{2}Z_T^{-1}\left[\frac{z}{z-4}\right]$$

$$= \frac{1}{2}\cdot 2^n + 2^n\cdot n^2 - \frac{1}{2}\cdot 4^n$$

Thus $\mathbf{Z_T^{-1}[\bar{u}(z)] = u_n = 2^{n-1} + 2^n \cdot n^2 - 2^{2n-1}}$

Type-2. Power series Method

Working procedure for problems

➲ The given $\bar{u}(z)$ is modified into a suitable form so as to expand it as an infinite series by using some of the standard expansions like

(i) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$

(ii) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

(iii) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

➲ $\bar{u}(z)$ is put in the form $\sum_{n=0}^{\infty} u_n z^{-n}$

That is $\bar{u}(z) = Z_T(u_n)$

➲ The required $Z_T^{-1}[\bar{u}(z)]$ will be u_n

WORKED PROBLEMS

33. *Find the inverse Z-transform of*

$$(a)\ \log\left(\frac{z}{z+1}\right) \quad (b)\ z\log\left(\frac{z}{z+1}\right)$$

>> **(a)** Let $\bar{u}(z) = \log\left(\frac{z}{z+1}\right)$

ie., $\bar{u}(z) = \log\left[\frac{z}{z\left(1+\frac{1}{z}\right)}\right] = -\log\left(1+\frac{1}{z}\right)$

We have $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$\therefore$ $\bar{u}(z) = -\left\{\left(\frac{1}{z}\right) - \frac{1}{2}\left(\frac{1}{z^2}\right) + \frac{1}{3}\left(\frac{1}{z^3}\right) - \frac{1}{4}\left(\frac{1}{z^4}\right) + \cdots\right\}$

ie., $\bar{u}(z) = -\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z^2}\right) - \frac{1}{3}\left(\frac{1}{z^3}\right) + \frac{1}{4}\left(\frac{1}{z^4}\right) - \cdots$

ie., $\bar{u}(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\frac{1}{z^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{-n} = \sum_{n=1}^{\infty} u_n z^{-n}$

Thus $u_n = Z_T^{-1}[\bar{u}(z)] = \begin{cases} \dfrac{(-1)^n}{n} & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}$

(b) Let $\bar{u}(z) = z\log\left(\dfrac{z}{z+1}\right)$

$$\bar{u}(z) = z\left\{-\frac{1}{z}+\frac{1}{2}\left(\frac{1}{z^2}\right)-\frac{1}{3}\left(\frac{1}{z^3}\right)+\frac{1}{4}\left(\frac{1}{z^4}\right)-\cdots\right\}$$

$$= -1+\frac{1}{2}\left(\frac{1}{z}\right)-\frac{1}{3}\left(\frac{1}{z^2}\right)+\frac{1}{4}\left(\frac{1}{z^3}\right)-\cdots\cdot$$

$$\bar{u}(z) = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+1}z^{-n} = \sum_{n=0}^{\infty}u_n z^{-n}$$

Thus $u_n = Z_T^{-1}[\bar{u}(z)] = \dfrac{(-1)^{n+1}}{n+1}$

---

34. *Obtain the inverse Z-transform of* $\dfrac{z}{(z+1)^2}$ *by power series expansion*

>> Let $\bar{u}(z) = \dfrac{z}{(z+1)^2}$

ie., $\bar{u}(z) = \dfrac{z}{z^2\left(1+\dfrac{1}{z}\right)^2} = \dfrac{1}{z}\left(1+\dfrac{1}{z}\right)^{-2}$

We have, $(1+x)^n = 1+nx+\dfrac{n(n-1)}{2!}x^2+\cdots$

$\therefore \quad (1+x)^{-2} = 1-2x+3x^2-4x^3+\cdots$

Now, $\bar{u}(z) = \dfrac{1}{z}\left\{1-2\cdot\dfrac{1}{z}+3\cdot\dfrac{1}{z^2}-4\cdot\dfrac{1}{z^3}+\cdots\right\}$

$$\bar{u}(z) = \frac{1}{z}-2\frac{1}{z^2}+3\frac{1}{z^3}-4\frac{1}{z^4}+\cdots$$

ie., $\bar{u}(z) = \displaystyle\sum_{n=0}^{\infty}(-1)^{n-1}n z^{-n} = \sum_{n=0}^{\infty}u_n z^{-n}$

Thus $u_n = Z_T^{-1}[\bar{u}(z)] = (-1)^{n-1}n$

---

35. *Find the inverse Z-transform of* $\dfrac{z^2}{(z-k)^2}$ *by power series method.*

>> Let $\overline{u}(z) = \dfrac{z^2}{(z-k)^2}$

ie., $\overline{u}(z) = \dfrac{z^2}{z^2\left(1-\dfrac{k}{z}\right)^2} = \left(1-\dfrac{k}{z}\right)^{-2}$

We have $(1-x)^{-2} = 1+2x+3x^2+4x^3+\cdots$

$\therefore$ $\overline{u}(z) = 1+2\left(\dfrac{k}{z}\right)+3\left(\dfrac{k}{z}\right)^2+4\left(\dfrac{k}{z}\right)^3+\cdots$

ie., $\overline{u}(z) = \sum_{n=0}^{\infty}(n+1)\left(\dfrac{k}{z}\right)^n = \sum_{n=0}^{\infty}[(n+1)\,k^n]\,z^{-n} = \sum_{n=0}^{\infty}u_n z^{-n}$

Thus $\boldsymbol{u_n = Z_T^{-1}[\overline{u}(z)] = (n+1)\,k^n}$

---

36. *Prove the following.* *(a)* $Z_T^{-1}[e^{1/z}] = \dfrac{1}{n!}$ *(b)* $Z_T^{-1}[z(e^{1/z}-1)] = \dfrac{1}{(n+1)!}$

>> **(a)** $e^{1/z} = 1 + \dfrac{1}{1!}\left(\dfrac{1}{z}\right) + \dfrac{1}{2!}\dfrac{1}{z^2}+\dfrac{1}{3!}\dfrac{1}{z^3}+\cdots$

ie., $e^{1/z} = \sum_{n=0}^{\infty}\dfrac{1}{n!}z^{-n} = Z_T\left[\dfrac{1}{n!}\right]$

Thus $\boldsymbol{Z_T^{-1}[e^{1/z}] = \dfrac{1}{n!}}$

(b) $z(e^{1/z}-1) = z\left(\dfrac{1}{1!}\cdot\dfrac{1}{z}+\dfrac{1}{2!}\dfrac{1}{z^2}+\dfrac{1}{3!}\dfrac{1}{z^3}+\cdots\right)$

ie., $z(e^{1/z}-1) = \dfrac{1}{1!}+\dfrac{1}{2!}\dfrac{1}{z}+\dfrac{1}{3!}\dfrac{1}{z^2}+\cdots$

ie., $z(e^{1/z}-1) = \sum_{n=0}^{\infty}\dfrac{1}{(n+1)!}z^{-n} = Z_T\left[\dfrac{1}{(n+1)!}\right]$

Thus $\boldsymbol{Z_T^{-1}[z(e^{1/z}-1)] = \dfrac{1}{(n+1)!}}$

EXERCISES

Find the inverse Z-transform of the following functions

1. $\dfrac{2z^2+3z}{(z+2)(z-4)}$
2. $\dfrac{z^2}{z^2-(a+b)z+ab}$ where $a \neq b$
3. $\dfrac{z}{2z^2+z-3}$
4. $\dfrac{8z^2}{(2z-1)(4z-1)}$
5. $\dfrac{z^2}{z^2-7z+12}$
6. $\dfrac{5z^2-2z}{(z-1)^4}$
7. $\dfrac{3z}{z^2-3z+2}$
8. $\dfrac{z}{z^2+11z+24}$
9. Given $\dfrac{\bar{u}(z)}{z} = \dfrac{2}{z+1} - \dfrac{3}{z-4} + \dfrac{4}{z-2}$, find u_n
10. Find the inverse Z-transform of $\left(\dfrac{z}{z-2}\right)^2$ by power series method.

ANSWERS

1. $\frac{1}{6}(-2)^n + \frac{11}{6}(4)^n$
2. $\dfrac{a^{n+1}-b^{n+1}}{a-b}$, $a \neq b$, $a > b$
3. $\frac{1}{5}\{1-(-3/2)^n\}$
4. $2(1/2)^n - (1/4)^n$
5. $4^{n+1} - 3^{n+1}$
6. $\frac{n}{2}(n^2+2n-3)$
7. $3(2^n-1)$
8. $\frac{1}{5}\{(-3)^n-(-8)^n\}$
9. $2(-1)^n - 3(4)^n + 4(2)^n$
10. $(n+1)2^n$

8.10 Solution of Difference Equations using Z-Transforms

Working procedure for problems

- We take Z-transforms on both sides of the given difference equation.
- We use the known expressions for the Z-transform for terms like u_{n+2}, u_{n+1}
- We obtain $\bar{u}(z) = Z_T(u_n)$ as a function of z
- The required solution $u_n = Z_T^{-1}[\bar{u}(z)]$

Note-1. *If the initial values $u_0, u_1, \ldots$ are not given we get the* ***general solution*** *of the given difference equation. If the values $u_0, u_1, \ldots$ are given specifically, we use them in the expressions of Z-transforms of u_{n+2}, u_{n+1} and obtain $\bar{u}(z) = Z_T(u_n)$. Further, the solution $u_n = Z_T^{-1}[\bar{u}(z)]$ so obtained will be the* ***particular solution*** *of the given difference equation.*

Note-2. ***Remember the following results in both ways.***

[*Z-transform and Inverse Z-transform*]

1. $Z_T(1) = \dfrac{z}{z-1}$	**2.** $Z_T(k^n) = \dfrac{z}{z-k}$
3. $Z_T(n) = \dfrac{z}{(z-1)^2}$	**4.** $Z_T(k^n n) = \dfrac{kz}{(z-k)^2}$
5. $Z_T(n^2) = \dfrac{z^2+z}{(z-1)^3}$	**6.** $Z_T(k^n n^2) = \dfrac{kz^2+k^2 z}{(z-k)^3}$
7. $Z_T(n^3) = \dfrac{z^3+4z^2+4z}{(z-1)^4}$	**8.** $Z_T(k^n n^3) = \dfrac{kz^3+4k^2 z^2+k^3 z}{(z-k)^3}$
9. $Z_T[\sin(n\pi/2)] = \dfrac{z}{z^2+1}$	**10.** $Z_T[\cos(n\pi/2)] = \dfrac{z^2}{z^2+1}$

Note-3

1. $Z_T(u_{n+1}) = z[\bar{u}(z) - u_0]$
2. $Z_T(u_{n+2}) = z^2[\bar{u}(z) - u_0 - u_1 z^{-1}]$

---

37. *Solve by using Z-transforms :* $u_{n+2} - 5u_{n+1} + 6u_n = 0$

>>Taking Z-transforms on both sides of the given equation we have,

$Z_T(u_{n+2}) - 5\,Z_T(u_{n+1}) + 6\,Z_T(u_n) = Z_T(0)$

ie., $z^2[\bar{u}(z) - u_0 - u_1 z^{-1}] - 5z[\bar{u}(z) - u_0] - 6\bar{u}(z) = 0$

ie., $[z^2 - 5z + 6]\,\bar{u}(z) - u_0(z^2 - 5z) - u_1 z = 0$

ie., $[z^2 - 5z + 6]\,\bar{u}(z) = u_0(z^2 - 5z) + u_1 z$

or $\bar{u}(z) = u_0 \cdot \dfrac{z^2 - 5z}{z^2 - 5z + 6} + u_1 \cdot \dfrac{z}{z^2 - 5z + 6}$

$$\Rightarrow \quad Z_T^{-1}[\bar{u}(z)] = u_0 Z_T^{-1}\left[\frac{z^2-5z}{(z-2)(z-3)}\right] + u_1 Z_T^{-1}\left[\frac{z}{(z-2)(z-3)}\right] \quad \ldots(1)$$

Let $\quad p(z) = \dfrac{z^2-5z}{(z-2)(z-3)}$

Further, let $\dfrac{p(z)}{z} = \dfrac{z-5}{(z-2)(z-3)} = \dfrac{A}{z-2} + \dfrac{B}{z-3}$

or $\quad z-5 = A(z-3) + B(z-2)$

Put $z = 3: \ -2 = B(1) \quad \therefore \ B = -2$

Put $z = 2: \ -3 = A(-1) \quad \therefore \ A = 3$

Hence $p(z) = 3\cdot\dfrac{z}{z-2} - 2\dfrac{z}{z-3}$

$$\Rightarrow \quad Z_T^{-1}[p(z)] = 3Z_T^{-1}\left[\frac{z}{z-2}\right] - 2Z_T^{-1}\left[\frac{z}{z-3}\right]$$

ie., $\quad Z_T^{-1}\left[\dfrac{z^2-5z}{z^2-5z+6}\right] = 3\cdot 2^n - 2\cdot 3^n \quad \ldots(2)$

Next, let $q(z) = \dfrac{z}{(z-2)(z-3)}$

Further, let $\dfrac{q(z)}{z} = \dfrac{1}{(z-2)(z-3)} = \dfrac{C}{z-2} + \dfrac{D}{z-3}$

or $\quad 1 = C(z-3) + D(z-2)$

Put $z = 2: \ 1 = C(-1) \quad \therefore \ C = -1$

Put $z = 3: \ 1 = D(1) \quad \therefore \ D = 1$

Hence $q(z) = -\dfrac{z}{z-2} + \dfrac{z}{z-3}$

$$\Rightarrow \quad Z_T^{-1}[q(z)] = -Z_T^{-1}\left[\frac{z}{z-2}\right] + Z_T^{-1}\left[\frac{z}{z-3}\right]$$

ie., $\quad Z_T^{-1}\left[\dfrac{z}{z^2-5z+6}\right] = -2^n + 3^n \quad \ldots(3)$

Using (2) and (3) in (1) we have,

$$Z_T^{-1}[\bar{u}(z)] = u_0\,[3.2^n - 2.3^n] + u_1\,[-2^n + 3^n]$$

ie., $\quad u_n = (3u_0 - u_1)\,2^n + (-2u_0 + u_1)\,3^n$

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Let $\quad c_1 = 3u_0 - u_1$ and $c_2 = -2u_0 + u_1$

Thus $u_n = c_1 \cdot 2^n + c_2 \cdot 3^n$, where c_1 and c_2 are arbitrary constants **is the general solution of the given difference equation.**

---

38. *Solve the difference equation $u_{n+2} + u_n = 0$ by using Z-transforms.*

>> Taking Z-transforms on both sides of the given equation we have,

$$Z_T(u_{n+2}) + Z_T(u_n) = Z_T(0)$$

ie., $\quad z^2[\bar{u}(z) - u_0 - u_1 z^{-1}] + \bar{u}(z) = 0$

ie., $\quad [z^2 + 1]\bar{u}(z) = u_0 z^2 + u_1 z$

or $\quad \bar{u}(z) = u_0 \cdot \dfrac{z^2}{z^2+1} + u_1 \cdot \dfrac{z}{z^2+1}$

$\Rightarrow \quad Z_T^{-1}[\bar{u}(z)] = u_0 Z_T^{-1}\left[\dfrac{z^2}{z^2+1}\right] + u_1 Z_T^{-1}\left[\dfrac{z}{z^2+1}\right]$

Thus $\quad \boldsymbol{u_n = u_0 \cos(n\pi/2) + u_1 \sin(n\pi/2)}$

where u_0 and u_1 are arbitrary constants is the required general solution of the given difference equation.

---

39. *Solve by using Z-transforms : $y_{n+2} - 4y_n = 0$, given that $y_0 = 0$ and $y_1 = 2$*

>> Taking Z-transforms on both sides of the given equation we have,

$$Z_T(y_{n+2}) - 4Z_T(y_n) = Z_T(0)$$

ie., $\quad z^2[\bar{y}(z) - y_0 - y_1 z^{-1}] - 4\bar{y}(z) = 0$

ie., $\quad [z^2 - 4]\bar{y}(z) - 2z = 0$, by using the given values.

or $\quad \bar{y}(z) = \dfrac{2z}{z^2-4}$

Let $\quad \dfrac{\bar{y}(z)}{z} = \dfrac{2}{(z-2)(z+2)} = \dfrac{A}{z-2} + \dfrac{B}{z+2}$

or $\quad 2 = A(z+2) + B(z-2)$

Put $z = 2$: $2 = A(4)$ $\quad \therefore \quad A = 1/2$

Put $z = -2$: $2 = B(-4)$ $\quad \therefore \quad B = -1/2$

Hence, $\bar{y}(z) = \frac{1}{2}\frac{z}{z-2} - \frac{1}{2}\frac{z}{z+2}$

$$\Rightarrow \quad Z_T^{-1}[\bar{y}(z)] = \frac{1}{2}\left\{Z_T^{-1}\left[\frac{z}{z-2}\right] - Z_T^{-1}\left[\frac{z}{z+2}\right]\right\}$$

ie., $$y_n = \frac{1}{2}\{2^n - (-2)^n\} = \frac{2^n}{2} + \frac{(-2)^n}{-2}$$

Thus $\boldsymbol{y_n = 2^{n-1} + (-2)^{n-1}}$ is the required particular solution.

---

40. *Solve by using Z-transforms :* $y_{n+1} + \frac{1}{4}y_n = \left(\frac{1}{4}\right)^n \; (n \geq 0),\; y_0 = 0$

>> Taking Z-transforms on both sides of the given equation we have,

$$Z_T(y_{n+1}) + \frac{1}{4}Z_T(y_n) = Z_T\left[\left(\frac{1}{4}\right)^n\right]$$

ie., $$z[\bar{y}(z) - y_0] + \frac{1}{4}\bar{y}(z) = \frac{z}{z-\frac{1}{4}}$$

ie., $$\left[z+\frac{1}{4}\right]\bar{y}(z) = \frac{z}{z-\frac{1}{4}},$$ by using the given value.

or $$\bar{y}(z) = \frac{z}{\left(z-\frac{1}{4}\right)\left(z+\frac{1}{4}\right)}$$

Let $$\frac{\bar{y}(z)}{z} = \frac{1}{\left(z-\frac{1}{4}\right)\left(z+\frac{1}{4}\right)} = \frac{A}{z-\frac{1}{4}} + \frac{B}{z+\frac{1}{4}}$$

or $$1 = A\left(z+\frac{1}{4}\right) + B\left(z-\frac{1}{4}\right)$$

Put $z = 1/4$: $1 = A(1/2)$ $\quad \therefore\; A = 2$

Put $z = -1/4$: $1 = B(-1/2)$ $\quad \therefore\; B = -2$

Hence, $$\bar{y}(z) = 2\cdot\frac{z}{z-\frac{1}{4}} - 2\cdot\frac{z}{z+\frac{1}{4}}$$

$$\Rightarrow \quad Z_T^{-1}[\bar{y}(z)] = 2\left\{Z_T^{-1}\left[\frac{z}{z-\frac{1}{4}}\right] - Z_T^{-1}\left[\frac{z}{z+\frac{1}{4}}\right]\right\}$$

Thus, $y_n = 2\left\{\left(\frac{1}{4}\right)^n - \left(-\frac{1}{4}\right)^n\right\}$ is the required particular solution.

---

41. *Solve :* $u_{n+2} - 3u_{n+1} + 2u_n = 1$ *by using Z-transforms.*

>> Taking Z-transforms on both sides of the given equation we have,

$$Z_T(u_{n+2}) - 3Z_T(u_{n+1}) + 2Z_T(u_n) = Z_T(1)$$

ie., $$z^2[\bar{u}(z) - u_0 - u_1 z^{-1}] - 3z[\bar{u}(z) - u_0] + 2\bar{u}(z) = \frac{z}{z-1}$$

ie., $$[z^2 - 3z + 2]\bar{u}(z) - u_0(z^2 - 3z) - u_1 z = \frac{z}{z-1}$$

ie., $$[(z-1)(z-2)]\bar{u}(z) = u_0(z^2 - 3z) + u_1 z + \frac{z}{z-1}$$

or $$\bar{u}(z) = u_0 \cdot \frac{z^2 - 3z}{(z-1)(z-2)} + u_1 \cdot \frac{z}{(z-1)(z-2)} + \frac{z}{(z-1)^2(z-2)}$$

ie., $$\bar{u}(z) = u_0 \cdot p(z) + u_1 \cdot q(z) + r(z) \ (say) \qquad \ldots (1)$$

We shall find the inverse Z-transform of $p(z)$, $q(z)$ and $r(z)$

Consider, $p(z) = \frac{z^2 - 3z}{(z-1)(z-2)}$

Let $$\frac{p(z)}{z} = \frac{z-3}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

or $z - 3 = A(z-2) + B(z-1)$

Put $z = 1 : -2 = A(-1) \quad \therefore A = 2$

Put $z = 2 : -1 = B(1) \quad \therefore B = -1$

$$\therefore \quad Z_T^{-1}[p(z)] = 2Z_T^{-1}\left[\frac{z}{z-1}\right] - Z_T^{-1}\left[\frac{z}{z-2}\right]$$

ie., $$Z_T^{-1}[p(z)] = 2.1 - 2^n = 2 - 2^n \qquad \ldots (2)$$

Consider $q(z) = \dfrac{z}{(z-1)(z-2)}$ (*Refer Problem-26*)

$$Z_T^{-1}[q(z)] = 2^n - 1 \quad \ldots(3)$$

Consider $r(z) = \dfrac{z}{(z-1)^2(z-2)}$

Let $\dfrac{z}{(z-1)^2(z-2)} = C \cdot \dfrac{z}{z-1} + D \cdot \dfrac{z}{(z-1)^2} + E \cdot \dfrac{z}{z-2}$

or $\quad 1 = C(z-1)(z-2) + D(z-2) + E(z-1)^2$

Put $z = 1 : \; 1 = D(-1) \quad \therefore \; D = -1$

Put $z = 2 : \; 1 = E(1) \quad \therefore \; E = 1$

Equating the coefficient of z^2 on both sides we get,

$C + E = 0 \quad \therefore \; C = -1$

Now, $Z_T^{-1}[r(z)] = -Z_T^{-1}\left[\dfrac{z}{z-1}\right] - Z_T^{-1}\left[\dfrac{z}{(z-1)^2}\right] + Z_T^{-1}\left[\dfrac{z}{z-2}\right]$

ie., $\quad Z_T^{-1}[r(z)] = -1 - n + 2^n \quad \ldots(4)$

With reference to (1) we have,

$$Z_T^{-1}[u(z)] = u_0 \cdot Z_T^{-1}[p(z)] + u_1 Z_T^{-1}[q(z)] + Z_T^{-1}[r(z)]$$

Using (2), (3) and (4) in the R.H.S we have,

$$Z_T^{-1}[\bar{u}(z)] = u_0(2 - 2^n) + u_1(2^n - 1) - 1 - n + 2^n$$

ie., $\quad u_n = (2u_0 - u_1 - 1) + (-u_0 + u_1 + 1)2^n - n$

Let us denote $c_1 = 2u_0 - u_1 - 1$ and $c_2 = -u_0 + u_1 + 1$ where c_1 and c_2 are arbitrary constants.

Thus $\boldsymbol{u_n = c_1 + c_2 \cdot 2^n - n}$ is the required solution.

---

42. *Solve by using Z-transforms :* $y_{n+2} + 2y_{n+1} + y_n = n$ *with* $y_0 = 0 = y_1$

>> Taking Z-transforms on both sides of the given equation we have,

$$Z_T(y_{n+2}) + 2Z_T(y_{n+1}) + Z_T(y_n) = Z_T(n)$$

ie., $\quad z^2[\bar{y}(z) - y_0 - y_1 z^{-1}] + 2z[\bar{y}(z) - y_0] + \bar{y}(z) = \dfrac{z}{(z-1)^2}$

ie., $[z^2 + 2z + 1]\,\bar{y}(z) = \dfrac{z}{(z-1)^2}$, by using the given values.

or $\bar{y}(z) = \dfrac{z}{(z-1)^2(z+1)^2}$

Let $\dfrac{z}{(z-1)^2(z+1)^2} = A\cdot\dfrac{z}{z-1} + B\cdot\dfrac{z}{(z-1)^2} + C\cdot\dfrac{z}{(z+1)} + D\cdot\dfrac{z}{(z+1)^2}$ ... (1)

or $1 = A(z-1)(z+1)^2 + B(z+1)^2 + C(z-1)^2(z+1) + D(z-1)^2$

Put $z = 1$: $1 = B(4)$ $\therefore$ $B = 1/4$

Put $z = -1$: $1 = D(4)$ $\therefore$ $D = 1/4$

Equating the coefficient of z^3 on both sides we get,

$A + C = 0$ or $C = -A$

Put $z = 0$: $1 = -A + B + C + D$

ie., $1 = C + 1/4 + C + 1/4$ or $1/2 = 2C$ $\therefore$ $C = 1/4$

Substituting $A = -1/4$, $B = C = D = 1/4$ in (1) and taking the inverse Z-transform we have,

$$Z_T^{-1}[\bar{y}(z)] = -\frac{1}{4}Z_T^{-1}\left[\frac{z}{z-1}\right] + \frac{1}{4}Z_T^{-1}\left[\frac{z}{(z-1)^2}\right] + \frac{1}{4}Z_T^{-1}\left[\frac{z}{z+1}\right] + \frac{1}{4}Z_T^{-1}\left[\frac{z}{(z+1)^2}\right]$$

ie., $y_n = -\dfrac{1}{4}\cdot 1 + \dfrac{1}{4}n + \dfrac{1}{4}(-1)^n + \dfrac{1}{4}\cdot(-1)(-1)^n n$

$= \dfrac{1}{4}[(n-1) - (-1)^n(n-1)]$

Thus $y_n = \dfrac{(n-1)}{4}[1 - (-1)^n]$ is the required solution.

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43. *Solve by using Z-transforms :* $u_{n+2} - 5u_{n+1} - 6u_n = 2^n$

>> Taking Z-transforms on both sides of the given equation we have,

$$Z_T(u_{n+2}) - 5Z_T(u_{n+1}) - 6Z_T(u_n) = Z_T(2^n)$$

ie., $z^2[\bar{u}(z) - u_0 - u_1 z^{-1}] - 5z[\bar{u}(z) - u_0] - 6\bar{u}(z) = \dfrac{z}{z-2}$

ie., $[z^2 - 5z - 6]\bar{u}(z) - u_0(z^2 - 5z) - u_1 z = \dfrac{z}{z-2}$

ie., $(z-6)(z+1)\bar{u}(z) = u_0(z^2 - 5z) + u_1 z + \dfrac{z}{z-2}$

or $\bar{u}(z) = u_0 \cdot \dfrac{z^2 - 5z}{(z-6)(z+1)} + u_1 \cdot \dfrac{z}{(z-6)(z+1)} + \dfrac{z}{(z-2)(z-6)(z+1)}$

ie., $\bar{u}(z) = u_0 \cdot p(z) + u_1 q(z) + r(z)$ (*say*) ...(1)

We shall find the inverse Z-transform of $p(z)$, $q(z)$ and $r(z)$.

Consider $p(z) = \dfrac{z^2 - 5z}{(z-6)(z+1)}$

Let $\dfrac{p(z)}{z} = \dfrac{z-5}{(z-6)(z+1)} = \dfrac{A}{z-6} + \dfrac{B}{z+1}$ or $z - 5 = A(z+1) + B(z-6)$

Put $z = 6$: $1 = A(7)$ $\therefore$ $A = 1/7$

Put $z = -1$: $-6 = B(-7)$ $\therefore$ $B = 6/7$

Hence $Z_T^{-1}[p(z)] = \dfrac{1}{7} Z_T^{-1}\left[\dfrac{z}{z-6}\right] + \dfrac{6}{7} Z_T^{-1}\left[\dfrac{z}{z+1}\right]$

ie., $Z_T^{-1}[p(z)] = \dfrac{1}{7}(6)^n + \dfrac{6}{7}(-1)^n$ ...(2)

Consider $q(z) = \dfrac{z}{(z-6)(z+1)}$

Let $\dfrac{q(z)}{z} = \dfrac{1}{(z-6)(z+1)} = \dfrac{C}{z-6} + \dfrac{D}{z+1}$

or $1 = C(z+1) + D(z-6)$

Put $z = 6$: $1 = C(7)$ $\therefore$ $C = 1/7$

Put $z = -1$: $1 = D(-7)$ $\therefore$ $D = -1/7$

Hence $Z_T^{-1}[q(z)] = \dfrac{1}{7} Z_T^{-1}\left[\dfrac{z}{z-6}\right] - \dfrac{1}{7} Z_T^{-1}\left[\dfrac{z}{z+1}\right]$

ie., $Z_T^{-1}[q(z)] = \dfrac{1}{7}(6)^n - \dfrac{1}{7}(-1)^n$ ...(3)

Consider $r(z) = \dfrac{z}{(z-2)(z-6)(z+1)}$

Let $\dfrac{r(z)}{z} = \dfrac{1}{(z-2)(z-6)(z+1)} = \dfrac{E}{z-2} + \dfrac{F}{z-6} + \dfrac{G}{z+1}$

or $\quad 1 = E(z-6)(z+1) + F(z-2)(z+1) + G(z-2)(z-6)$

Put $\quad z = 2 : \quad 1 = E(-12) \quad \therefore \; E = -1/12$

Put $\quad z = 6 : \quad 1 = F(28) \quad \therefore \; F = 1/28$

Put $\quad z = -1 : \quad 1 = G(21) \quad \therefore \; G = 1/21$

Hence $Z_T^{-1}[r(z)] = \dfrac{-1}{12} Z_T^{-1}\left[\dfrac{z}{z-2}\right] + \dfrac{1}{28} Z_T^{-1}\left[\dfrac{z}{z-6}\right] + \dfrac{1}{21} Z_T^{-1}\left[\dfrac{z}{z+1}\right]$

ie., $\quad Z_T^{-1}[r(z)] = \dfrac{-1}{12}(2)^n + \dfrac{1}{28}(6)^n + \dfrac{1}{21}(-1)^n \quad \ldots (4)$

With reference to (1) we have,

$$Z_T^{-1}[\bar{u}(z)] = u_0 Z_T^{-1}[p(z)] + u_1 \cdot Z_T^{-1}[q(z)] + Z_T^{-1}[r(z)]$$

Using (2), (3) and (4) in the R.H.S we have,

$$Z_T^{-1}[\bar{u}(z)] = u_0\left\{\frac{1}{7}(6)^n + \frac{6}{7}(-1)^n\right\} + u_1\left\{\frac{1}{7}(6)^n - \frac{1}{7}(-1)^n\right\} + \left\{\frac{-1}{12}(2)^n + \frac{1}{28}(6)^n + \frac{1}{21}(-1)^n\right\}$$

$$Z_T^{-1}[\bar{u}(z)] = u_n = \left[\frac{u_0}{7} + \frac{u_1}{7} + \frac{1}{28}\right](6)^n + \left[\frac{6u_0}{7} - \frac{u_1}{7} + \frac{1}{21}\right](-1)^n - \frac{2^n}{12}$$

Let us denote $c_1 = u_0/7 + u_1/7 + 1/28$ and $c_2 = 6u_0/7 - u_1/7 + 1/21$,

where c_1 and c_2 are arbitrary constants.

Thus $\quad \mathbf{u_n = c_1(6)^n + c_2(-1)^n - \dfrac{2^n}{12}}$ is the required solution.

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44. *Solve the difference equation* $y_{n+2} - 6y_{n+1} + 9y_n = 3^n$ *by using Z-transforms.*

>> Taking Z-transforms on both sides of the given equation we have,

$$Z_T(y_{n+2}) - 6 Z_T(y_{n+1}) + 9 Z_T(y_n) = Z_T(3^n)$$

ie., $\quad z^2[\bar{y}(z) - y_0 - y_1 z^{-1}] - 6z[\bar{y}(z) - y_0] + 9\bar{y}(z) = \dfrac{z}{z-3}$

ie., $\quad [z^2 - 6z + 9]\bar{y}(z) - y_0(z^2 - 6z) - y_1 z = \dfrac{z}{z-3}$

ie., $\quad (z-3)^2 \cdot \bar{y}(z) = y_0(z^2 - 6z) + y_1 z + \dfrac{z}{z-3}$

or $\quad \bar{y}(z) = y_0 \cdot \dfrac{(z^2 - 6z)}{(z-3)^2} + y_1 \dfrac{z}{(z-3)^2} + \dfrac{z}{(z-3)^3}$

ie., $\quad \bar{y}(z) = y_0 \cdot p(z) + y_1 \cdot q(z) + r(z)$ (*say*) $\quad \ldots (1)$

We shall find the inverse Z-transforms of $p(z)$, $q(z)$ and $r(z)$

Consider $p(z) = \dfrac{z^2 - 6z}{(z-3)^2}$

We take note that $Z_T(3^n) = \dfrac{z}{z-3}$ and $Z_T(3^n \cdot n) = \dfrac{3z}{(z-3)^2}$

Let $\quad p(z) = \dfrac{z^2 - 6z}{(z-3)^2} = A \cdot \dfrac{z}{z-3} + B \cdot \dfrac{3z}{(z-3)^2}$

or $\quad z - 6 = A(z-3) + 3B$

Put $\quad z = 3 : \; -3 = 3B \quad \therefore \; B = -1$

Also, $\quad -3A + 3B = -6 \quad \therefore \; A = 1$

Hence $Z_T^{-1}[p(z)] = Z_T^{-1}\left[\dfrac{z}{z-3}\right] - Z_T^{-1}\left[\dfrac{3z}{(z-3)^2}\right]$

ie., $\quad Z_T^{-1}[p(z)] = 3^n - 3^n \cdot n \quad \ldots (2)$

Consider $q(z) = \dfrac{z}{(z-3)^2} = \dfrac{1}{3} \cdot \dfrac{3z}{(z-3)^2}$

$\Rightarrow \quad Z_T^{-1}[q(z)] = \dfrac{1}{3} Z_T^{-1}\left[\dfrac{3z}{(z-3)^2}\right]$

ie., $\quad Z_T^{-1}[q(z)] = \dfrac{1}{3}(3^n n) \quad \ldots (3)$

Consider $r(z) = \dfrac{z}{(z-3)^3}$

We take note that $Z_T(3^n n^2) = \dfrac{3z^2 + 9z}{(z-3)^3}$

Let $\quad r(z) = \dfrac{z}{(z-3)^3} = C \cdot \dfrac{z}{z-3} + D \cdot \dfrac{3z}{(z-3)^2} + E \cdot \dfrac{3z^2 + 9z}{(z-3)^3}$

or $\quad 1 = C(z-3)^2 + 3D(z-3) + E(3z+9)$

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Put $z = 3 : \ 1 = E(18) \qquad \therefore \quad E = 1/18$

Equating the coefficient of z^2 and z on both sides we get,

$$C = 0 \quad \text{and} \quad -6C + 3D + 3E = 0 \quad \therefore \quad D = -1/18$$

Now, $$Z_T^{-1}[r(z)] = -\frac{1}{18} Z_T^{-1}\left[\frac{3z}{(z-3)^2}\right] + \frac{1}{18} Z_T^{-1}\left[\frac{3z^2+9z}{(z-3)^3}\right]$$

ie., $$Z_T^{-1}[r(z)] = -\frac{1}{18}(3^n \cdot n) + \frac{1}{18}(3^n \cdot n^2) \qquad \ldots (4)$$

With reference to (1) we have,

$$Z_T^{-1}[\bar{y}(z)] = y_0 Z_T^{-1}[p(z)] + y_1 Z_T^{-1}[q(z)] + Z_T^{-1}[r(z)]$$

Using (2), (3) and (4) in the R.H.S we have,

$$y_n = y_0(3^n - 3^n \cdot n) + y_1 \cdot \frac{1}{3}(3^n n) - \frac{1}{18}(3^n \cdot n) + \frac{1}{18}(3^n \cdot n^2)$$

ie., $$y_n = y_0 3^n + \left(-y_0 + \frac{1}{3} y_1\right)(3^n n) + \frac{3^n}{18}(n^2 - n)$$

We denote $c_1 = y_0$ and $c_2 = -y_0 + (y_1/3)$, c_1 and c_2 are arbitrary constants.

$$\therefore \quad y_n = c_1(3^n) + c_2(3^n n) + \frac{3^n}{3^2 \cdot 2} n(n-1)$$

Thus $\boldsymbol{y_n = (c_1 + c_2 n) 3^n + \frac{3^{n-2}}{2} n(n-1)}$ is the required solution.

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45. *Solve the difference equation,* $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ *with* $y_0 = y_1 = 0$ *using Z-transforms.*

>> Taking Z-transforms on both sides of the given equation we have,

$$Z_T(y_{n+2}) + 6Z_T(y_{n+1}) + 9Z_T(y_n) = Z_T(2^n)$$

ie., $$z^2[\bar{y}(z) - y_0 - y_1 z^{-1}] + 6z[\bar{y}(z) - y_o] + 9\bar{y}(z) = \frac{z}{z-2}$$

ie., $$[z^2 + 6z + 9]\,\bar{y}(z) = \frac{z}{z-2}, \text{ by using the initial values.}$$

or $$\bar{y}(z) = \frac{z}{(z-2)(z+3)^2}$$

Let $\quad \dfrac{z}{(z-2)(z+3)^2} = A\cdot\dfrac{z}{z-2} + B\cdot\dfrac{z}{z+3} + C\cdot\dfrac{z}{(z+3)^2}$

or $\quad 1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2)$

Put $\quad z = 2 \;:\; 1 = A(25) \;\therefore\; A = 1/25$

Put $\quad z = -3 \;:\; 1 = C(-5) \;\therefore\; C = -1/5$

Equating the coefficient of z^2 on both sides we get, $0 = A + B \quad \therefore \quad B = -1/25$

Hence $\quad \bar{y}(z) = \dfrac{1}{25}\cdot\dfrac{z}{z-2} - \dfrac{1}{25}\,\dfrac{z}{z+3} - \dfrac{1}{5}\cdot\dfrac{z}{(z+3)^2}$

or $\quad \bar{y}(z) = \dfrac{1}{25}\cdot\dfrac{z}{z-2} - \dfrac{1}{25}\cdot\dfrac{z}{z+3} - \dfrac{1}{5}\cdot\dfrac{1}{-3}\cdot\dfrac{-3z}{(z+3)^2}$

$\Rightarrow \quad Z_T^{-1}[\bar{y}(z)] = \dfrac{1}{25}Z_T^{-1}\left[\dfrac{z}{z-2}\right] - \dfrac{1}{25}\;Z_T^{-1}\left[\dfrac{z}{z+3}\right] + \dfrac{1}{15}Z_T^{-1}\left[\dfrac{-3z}{(z+3)^2}\right]$

ie., $\quad y_n = \dfrac{1}{25}(2)^n - \dfrac{1}{25}(-3)^n + \dfrac{1}{15}(-3)^n\cdot n$

Thus $\quad y_n = \dfrac{1}{5}\left\{\dfrac{1}{5}(2)^n - \dfrac{1}{5}(-3)^n + \dfrac{1}{3}(-3)^n\cdot n\right\}$ is the required solution.

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46. *The equation $u_{n+2} - 2u_{n+1} + u_n = 3n+5$ is satisfied by a sequence u_n. Find this sequence using Z-transforms*

>> Taking Z-transforms on both sides of the given equation we have,

$$Z_T(u_{n+2}) - 2Z_T(u_{n+1}) + Z_T(u_n) = 3Z_T(n) + 5Z_T(1)$$

ie., $\quad z^2[\bar{u}(z) - u_0 - u_1 z^{-1}] - 2z[\bar{u}(z) - u_0] + \bar{u}(z) = 3\cdot\dfrac{z}{(z-1)^2} + 5\cdot\dfrac{z}{z-1}$

ie., $\quad [z^2 - 2z + 1]\,\bar{u}(z) - u_0(z^2 - 2z) - u_1 z = 3\dfrac{z}{(z-1)^2} + 5\dfrac{z}{(z-1)}$

ie., $\quad (z-1)^2\,\bar{u}(z) = u_0(z^2-2z) + u_1 z + 3\dfrac{z}{(z-1)^2} + 5\dfrac{z}{(z-1)}$

or $\quad \bar{u}(z) = u_0\cdot\dfrac{z^2-2z}{(z-1)^2} + u_1\cdot\dfrac{z}{(z-1)^2} + 3\dfrac{z}{(z-1)^4} + 5\dfrac{z}{(z-1)^3}$

or $\quad \bar{u}(z) = u_0\cdot\dfrac{z^2-2z}{(z-1)^2} + u_1\cdot\dfrac{z}{(z-1)^2} + \dfrac{5z^2-2z}{(z-1)^4}$

ie., $\bar{u}(z) = u_0 \cdot p(z) + u_1 \cdot q(z) + r(z)$ (*say*) . . . (1)

We shall find the inverse Z-transforms of $p(z)$, $q(z)$ and $r(z)$

Consider $p(z) = \dfrac{z^2 - 2z}{(z-1)^2} = A \cdot \dfrac{z}{z-1} + B \cdot \dfrac{z}{(z-1)^2}$

or $\quad z - 2 = A(z-1) + B$

$\Rightarrow \quad A = 1, -A + B = -2 \quad \therefore\ B = -1$

Hence $Z_T^{-1}[p(z)] = Z_T^{-1}\left[\dfrac{z}{z-1}\right] - Z_T^{-1}\left[\dfrac{z}{(z-1)^2}\right]$

ie., $\quad Z_T^{-1}[p(z)] = 1 - n$ . . . (2)

Also $\quad Z_T^{-1}[q(z)] = Z_T^{-1}\left[\dfrac{z}{(z-1)^2}\right] = n$ . . . (3)

Consider $r(z) = \dfrac{5z^2 - 2z}{(z-1)^4}$

We also take note that

$$Z_T^{-1}\left[\frac{z^2+z}{(z-1)^3}\right] = n^2 \text{ and } Z_T^{-1}\left[\frac{z^3+4z^2+z}{(z-1)^4}\right] = n^3$$

Let $\quad r(z) = \dfrac{5z^2 - 2z}{(z-1)^4} = C \cdot \dfrac{z}{z-1} + D \cdot \dfrac{z}{(z-1)^2} + E \cdot \dfrac{z^2+z}{(z-1)^3} + F \cdot \dfrac{z^3+4z^2+z}{(z-1)^4}$

or $\quad 5z - 2 = C(z-1)^3 + D(z-1)^2 + E(z+1)(z-1) + F(z^2+4z+1)$

Put $\quad z = 1 : \ 3 = F(6) \quad \therefore\ F = 1/2$

Equating the coefficients of z^3, z^2 and z on both sides we get,

$$C = 0, \ -3C + D + E + F = 0 \ \text{ and } \ 3C - 2D + 4F = 5$$

By solving we get, $D = -3/2$ and $E = 1$

Hence, $Z_T^{-1}[r(z)] = \dfrac{-3}{2} Z_T^{-1}\left[\dfrac{z}{(z-1)^2}\right]$

$$+ Z_T^{-1}\left[\frac{z^2+z}{(z-1)^3}\right] + \frac{1}{2} Z_T^{-1}\left[\frac{z^3+4z^2+4z}{(z-1)^4}\right]$$

ie., $\quad Z_T^{-1}[r(z)] = \dfrac{-3}{2} \cdot n + n^2 + \dfrac{1}{2} n^3$

ie., $Z_T^{-1}[r(z)] = \frac{n}{2}(n^2+2n-3)$ . . . (4)

With reference to (1), we have,

$$Z_T^{-1}[u(z)] = u_0 Z_T^{-1}[p(z)] + u_1 Z_T^{-1}[q(z)] + Z_T^{-1}[r(z)]$$

Using (2), (3) and (4) in the R.H.S, we have

$$Z_T^{-1}[\bar{u}(z)] = u_0(1-n) + u_1(n) + \frac{n}{2}(n^2+2n-3)$$

$$Z_T^{-1}[\bar{u}(z)] = u_n = u_0 + \left(-u_0 + u_1 - \frac{3}{2}\right)n + \frac{n(n^2+2n)}{2}$$

We denote $c_1 = u_0$ and $c_2 = -u_0 + u_1 - (3/2)$, c_1 and c_2 are arbitrary constants.

Thus $\boldsymbol{u_n = c_1 + c_2 n + \frac{n^2(n+2)}{2}}$ is the required sequence.

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47. *Find the response of the system $y_{n+2} - 4y_{n+1} + 3y_n = u_n$ with $y_0 = 0$, $y_1 = 1$ and $u_n = 1$ for $n = 0, 1, 2, 3, \ldots$ by Z-transform method.*

>> The given equation is $y_{n+2} - 4y_{n+1} + 3y_n = 1$.

Taking Z-transforms on both sides we have,

$$Z_T(y_{n+2}) - 4Z_T(y_{n+1}) + 3Z_T(y_n) = Z_T(1)$$

ie., $z^2[\bar{y}(z) - y_0 - y_1 z^{-1}] - 4z[\bar{y}(z) - y_0] + 3\bar{y}(z) = \frac{z}{z-1}$

ie., $[z^2 - 4z + 3]\bar{y}(z) - z = \frac{z}{z-1}$

ie., $(z-1)(z-3)\bar{y}(z) = z + \frac{z}{(z-1)}$

or $\bar{y}(z) = \frac{z}{(z-1)(z-3)} + \frac{z}{(z-1)^2(z-3)}$

ie., $\bar{y}(z) = p(z) + q(z)$ *(say)* . . . (1)

Consider $p(z) = \frac{z}{(z-1)(z-3)}$

Let $\frac{p(z)}{z} = \frac{1}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3}$

or $1 = A(z-3) + B(z-1)$

Put $z = 1$: $1 = A(-2)$ $\therefore$ $A = -1/2$

Put $z = 3 : \quad 1 = B(2) \qquad \therefore \; B = 1/2$

Hence $Z_T^{-1}[p(z)] = \frac{-1}{2} Z_T^{-1}\left[\frac{z}{z-1}\right] + \frac{1}{2} Z_T^{-1}\left[\frac{z}{z-3}\right]$

ie., $\quad Z_T^{-1}[p(z)] = \frac{-1}{2} \cdot 1 + \frac{1}{2}(3^n) \qquad \ldots (2)$

Consider $q(z) = \frac{z}{(z-1)^2 (z-3)}$

Let $\quad q(z) = \frac{z}{(z-1)^2(z-3)} = C \cdot \frac{z}{z-1} + D \cdot \frac{z}{(z-1)^2} + E \cdot \frac{z}{z-3}$

or $\quad 1 = C(z-1)(z-3) + D(z-3) + E(z-1)^2$

Put $\quad z = 1 : \quad 1 = D(-2) \qquad \therefore \; D = -1/2$

Put $\quad z = 3 : \quad 1 = E(4) \qquad \therefore \; E = 1/4$

Also we must have $C + E = 0 \qquad \therefore \; C = -1/4$

Hence $Z_T^{-1}[q(z)] = -\frac{1}{4} Z_T^{-1}\left[\frac{z}{z-1}\right] - \frac{1}{2} Z_T^{-1}\left[\frac{z}{(z-1)^2}\right] + \frac{1}{4} Z_T^{-1}\left[\frac{z}{z-3}\right]$

ie., $\quad Z_T^{-1}[q(z)] = \frac{-1}{4} \cdot 1 - \frac{1}{2} \cdot n + \frac{1}{4} \cdot 3^n \qquad \ldots (3)$

With reference to (1) we have,

$$Z_T^{-1}[\bar{y}(z)] = Z_T^{-1}[p(z)] + Z_T^{-1}[q(z)]$$

Using (2) and (3) in the R.H.S we have,

$$Z_T^{-1}[\bar{y}(z)] = -\frac{1}{2} + \frac{3^n}{2} - \frac{1}{4} - \frac{n}{2} + \frac{3^n}{4}$$

ie., $\quad y_n = -\frac{3}{4} + \frac{3}{4} \cdot 3^n - \frac{n}{2}$

Thus $\quad \boldsymbol{y_n = \frac{1}{4}[-3 + 3^{n+1} - 2n]}$ is the required response.

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48. *Find the impulse response of a system described by* $y_{n+1} + 2y_n = \delta_n \; ; \; y_0 = 0$ *by applying Z-transforms.*

>> Taking Z-transforms on both sides of the given equation we have,

$$Z_T(y_{n+1}) + 2Z_T(y_n) = Z_T(\delta_n)$$

ie., $\quad z[\bar{y}(z) - y_0] + 2\bar{y}(z) = 1$

ie., $(z+2)\bar{y}(z) = 1$ or $\bar{y}(z) = \dfrac{1}{z+2}$

ie., $\bar{y}(z) = \dfrac{1}{z\left(1+\dfrac{2}{z}\right)} = \dfrac{1}{z}\left(1+\dfrac{2}{z}\right)^{-1}$

Now, $\bar{y}(z) = \dfrac{1}{z}\left\{1-\dfrac{2}{z}+\left(\dfrac{2}{z}\right)^2-\left(\dfrac{2}{z}\right)^3+\cdots\right\}$

$$\bar{y}(z) = \frac{1}{z}-2\left(\frac{1}{z^2}\right)+2^2\left(\frac{1}{z^3}\right)-2^3\left(\frac{1}{z^4}\right)+\cdots$$

$$\bar{y}(z) = \sum_{n=1}^{\infty}(-1)^{n-1}2^{n-1}\left(\frac{1}{z^n}\right) = \sum_{n=1}^{\infty}(-1)^{n-1}\,2^{n-1}\,z^{-n}$$

ie., $Z_T(y_n) = \sum_{n=1}^{\infty}(-2)^{n-1}\,z^{-n}$

Thus $\mathbf{y_n = (-2)^{n-1}}$ where $n \geq 1$

EXERCISES

Solve the following difference equations using Z- transforms

1. $y_{n+2}-y_{n+1}-2y_n = 0$

2. $y_{n+2}-6y_{n+1}+9y_n = 0$

3. $y_{n+2}-9y_n = 0$ given that $y_0 = 0, y = 2$

4. $y_{n+2}-5y_{n+1}-6y_n = 4^n$

5. $y_{n+2}-2y_{n+1}+y_n = 2^n$; $y_0 = 2$ and $y_1 = 1$

6. $u_{n+2}-u_n = (n-1)$ given that $u_0 = 1$ and $u_1 = 2$

7. $u_{n+2}-4u_{n+1}+3u_n = 5^n$

8. $u_{n+2}+4u_{n+1}+3u_n = 3^n$ with $u_0 = 0$ and $u_1 = 1$

9. $u_{n+2}+6u_{n+1}+9u_n = 2^n$ with $u_0 = 0 = u_1$

10. Find the response of the system $y_{n+2}-5y_{n+1}+6y_n = u_n$ with $y_0 = 0$, $y_1 = 1$ and $u_n = 1$ for $n = 0, 1, 2, 3, \ldots$

ANSWERS

1. $y_n = c_1 (2)^n + c_2 (-1)^n$
2. $y_n = (c_1 + c_2 n) 3^n$
3. $y_n = 3^{n-1} + (-3)^{n-1}$
4. $y_n = c_1 (6)^n + c_2 (-1)^n - (4^n/10)$
5. $y_n = 1 - 2n + 2^n$
6. $u_n = \frac{1}{9} [9(2)^n - (-2)^n + 1 - 3^n]$
7. $u_n = c_1 + c_2 (3)^n + 5^n/8$
8. $u_n = \frac{1}{24} [9(-1)^n - 10(-3)^n + 3^n]$
9. $u_n = \frac{1}{25}\left[2^n - (-3)^n + \frac{5n}{3} (-3)^n \right]$
10. $u_n = \frac{1}{2} [1 - 2^{n+2} + 3^{n+1}]$

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